ON RESONANCES AND THE FORMATION OF GAPS IN THE SPECTRUM OF QUASI-PERIODIC SCHRÖDINGER EQUATIONS

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Abstract. We consider one-dimensional difference Schrödinger equations

$$\big[H(x,\omega)\varphi\big](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(x+n\omega)\varphi(n) = E\varphi(n)\ ,$$

$n \in \mathbb{Z}$, $x, \omega \in [0,1]$ with real-analytic potential function $V(x)$. If $L(E,\omega_0) > 0$ for all $E \in (E', E'')$ and some Diophantine ω_0, then the integrated density of states is absolutely continuous for almost every ω close to ω_0, see [GolSch2]. In this work we apply the methods and results of [GolSch2] to establish the formation of a dense set of gaps in $\mathrm{spec}(H(x,\omega)) \cap (E', E'')$. Our approach is based on an induction on scales argument, and is therefore both constructive as well as quantitative. Resonances between eigenfunctions of one scale lead to "pre-gaps" at a larger scale. To pass to actual gaps in the spectrum, we show that these pre-gaps cannot be filled more than a finite (and uniformly bounded) number of times. To accomplish this, one relates a pre-gap to pairs of complex zeros of the Dirichlet determinants off the unit circle using the techniques of [GolSch2]. Amongst other things, we establish in this work a non-perturbative version of the co-variant parametrization of the eigenvalues and eigenfunctions via the phases in the spirit of Sinai's (perturbative) description of the spectrum [Sin] via his function Λ. This allows us to relate the gaps in the spectrum with the graphs of the eigenvalues parametrized by the phase. Our infinite volume theorems hold for all Diophantine frequencies ω up to a set of Hausdorff dimension zero.

1. Introduction and statement of the main results

The main goal of this work is to establish a multiscale description of the structure of the spectrum of quasi-periodic Schrödinger equations

$$\big[H(x,\omega)\varphi\big](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(x+n\omega)\varphi(n) = E\varphi(n) \tag{1.1}$$

in the regime of exponentially localized eigenfunctions. We assume that $V(x)$ is a 1-periodic, real-analytic function, and that $\omega \in [0,1]$. Let $H_N(x,\omega)$ be the restriction of $H(x,\omega)$ to the finite interval $[1,N]$ with zero boundary conditions. Consider the union $\mathcal{S}_N = \bigcup\limits_x \mathrm{spec}(H_N(x,\omega))$, where $\mathrm{spec}(H_N(x,\omega))$ stands for the spectrum of $H_N(x,\omega)$. The set $\mathcal{S}_N$ is closed, so

$$\mathcal{S}_N = \big[\underline{E}(N), \overline{E}(N)\big] \setminus \bigcup_k \big(\underline{E}(N,k), \overline{E}(N,k)\big), \quad \underline{E}(N) = \min_{\mathcal{S}_N} E,\ \overline{E}(N) = \max_{\mathcal{S}_N} E$$

where $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ are the maximal intervals of $\big[\underline{E}(N), \overline{E}(N)\big] \setminus \mathcal{S}_N$. More specifically, the goals of this work are as follows:

(a) To relate the intervals $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ and $\big(\underline{E}(N',k'), \overline{E}(N',k')\big)$ for "consecutive scales" $N \gg N'$.

(b) To "label" the interval $\big(\underline{E}(N,k),\ \overline{E}(N,k)\big)$ relative to the intervals $\big(\underline{E}(m,\ell), \overline{E}(m,\ell)\big)$ of the previous scales.

(c) To describe the mechanism responsible for the formation of the intervals $\big(\underline{E}(N,k), \overline{E}(N,k)\big)$ inside the set $\mathcal{S}_{N'}$, $N' \ll N$, independently of any $\big(\underline{E}(N',k'), \overline{E}(N',k')\big)$.

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Our interest in these properties is largely motivated by possible applications to inverse spectral problems for the quasi-periodic Schrödinger equation and the Toda lattice with quasi-periodic initial data [Tod]. This paper relies heavily on the methods developed in [GolSch2]. For the convenience of the reader, we recall – and expand upon – some of the material of that paper in Sections 2-7. The eigenvalues

$$E_1^{(N)}(x,\omega) < E_2^{(N)}(x,\omega) < \cdots < E_N^{(N)}(x,\omega)$$

of $H_N(x,\omega)$ are real analytic functions of $x \in [0,1]$. Although the graphs of the functions $E_j^{(N)}(x,\omega)$ can be very complicated, the following was proved in [GolSch2] for Diophantine ω, see (1.6), and positive Lyapunov exponents: there exist intervals $\big(E'_{N,k}, E''_{N,k}\big)$, $k = 1, 2, \ldots, k_N$, with

$$\max_k \big(E''_{N,k} - E'_{N,k}\big) \le \exp\big(-\big(\log N\big)^A\big), \qquad k_N \le \exp\big(\big(\log N\big)^B\big),$$

with constants $1 \ll B \ll A$ depending on ω, such that if

$$E_j^{(N)}(x,\omega) \notin \mathcal{E}_N = \bigcup_k \big(E'_{N,k}, E''_{N,k}\big),$$

for some j and x, then

$$\big|\partial_x E_j^{(N)}(x,\omega)\big| > \exp\big(-N^\delta\big)$$

Here $0 < \delta \ll 1$ is an arbitrary but fixed small parameter. In other words, the graphs of $E_j(x,\omega)$ have controlled slopes off a small set $\mathcal{E}_N$.

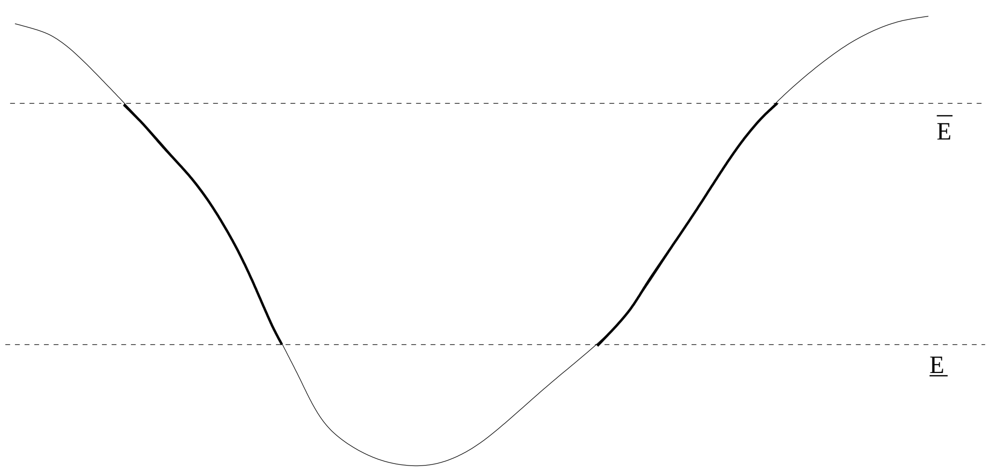

FIGURE 1. I–segments

The segments of the graph where $E_j^{(N)}(x,\omega) \in I$ and $I = (\underline{E}, \overline{E})$ is an interval disjoint from $\mathcal{E}_N$, are called *I–segments*. They are denoted by $\big\{E_j^{(N)}(x,\omega), \underline{x}, \bar{x}\big\}$, where

$$E_j^{(N)}(\underline{x},\omega) = \underline{E}, \quad E_j^{(N)}(\bar{x},\omega) = \overline{E}$$

The I–segments are important for our purposes, because they allow us to locate the resonances and to describe the graphs of the functions $E_j^{(\overline{N})}(x,\omega)$ for $\overline{N} \gg N$ in the region where the resonance occurs. A possible definition of a resonance is as follows: With a constant $A \gg 1$ depending on ω,

$$\tau = \big|E_{j_1}^{(N)}(x,\omega) - E_{j_2}^{(N)}(x + m\omega, \omega)\big| < m^{-A} \tag{1.2}$$

for some $x \in \mathbb{T}$, $1 \le j_1$, $j_2 \le N$ and $m > N$. In fact, there is some stability in the constant A with regard to small perturbations of ω.

The significance of such resonances was explained in the work by Sinai [Sin] on quasi-periodic Anderson localization for potentials $V(x) = \lambda\cos(2\pi x)$ in the regime of large $|\lambda|$, see (1.1). Sinai developed a KAM-type scheme to analyze the functions $E_j^{(N)}(x,\omega)$ and the corresponding eigenvectors. The critical points of $E_j^{(\overline{N})}(x,\omega)$ with $\overline{N} \gg N$ were proved to be closely related to resonances as in (1.2). It is very important for the analysis of the resonances (1.2) in [Sin] that given $x \in \mathbb{T}$ and j_1 there exist at most one j_2 and $m \le \overline{N}$ so that (1.2) holds. *For that reason the function $V(x)$ in* [Sin] *is assumed to have two monotonicity intervals with non-degenerate critical points.* That allows one to reduce the analysis of $E_j^{(\overline{N})}(x,\omega)$ to an eigenvalue problem for a 2×2 matrix function of the form

$$A(x) = \begin{bmatrix} E_1(x-x_0) & \varepsilon(x) \\ \varepsilon(x) & E_2(x-x_0) \end{bmatrix}, \tag{1.3}$$

where $E_1(0) = E_2(0)$, $\partial_x E_1 < 0$, $\partial_x E_2 > 0$ locally around zero, and $\varepsilon(x)$ is small together with its derivatives. It is easy to check that the eigenvalues $E^+(x)$, $E^-(x)$ of $A(x)$ plotted against x are as in Figure 2, at least locally around x_0.

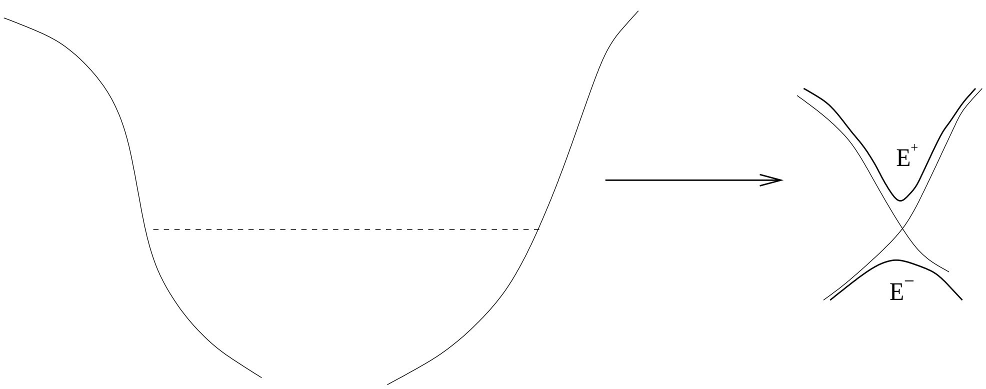

FIGURE 2. Classical formation of the resonant eigenvalues

We would like to emphasize that some of the conclusions which we reach in this paper are similar in spirit to those of Sinai [Sin]. This is particularly true in regards to the main result involving gaps and the aforementioned pictures describing the splitting of eigenvalues. At the same time, we stress that we use entirely nonperturbative methods (i.e., we are only assuming positive Lyapunov exponent rather than large $|\lambda|$) and we work with more general potentials than cosine. In this respect we would like to mention the recent breakthrough by Puig [Pui], who established the Cantor structure of the spectrum for the almost Mathieu case (cosine potential) and Diophantine ω. Earlier, Choi, Elliott, and Yui [ChoEllYui] had obtained gaps for the case of Liouville rotation numbers ω. The remaining cases of irrational rotation numbers (i.e., those with behavior intermediate to Diophantine and Liouville) was settled by Avila and Jitomirskaya [AviJit] (but this again only applies to the cosine).

The major objective in this work is to locate those segments of the graphs of some $E_{k_1}^{(\overline{N})}(x,\omega)$, $E_{k_2}^{(\overline{N})}(x,\omega)$ which look like $E^+(x,\omega)$ and $E^-(x,\omega)$ in Figure 2. Ultimately, such regions give rise to gaps in the spectrum. Before we state the main result of this work let us recall the central notions involved in it.

It is convenient to replace $V(x)$ in (1.1) by $V\big(e(x)\big)$ (with $e(x) = e^{2\pi i x}$), where $V(z)$ is an analytic function in the annulus $\mathcal{A}_{\rho_0} = \big\{z \in \mathbb{C} : 1-\rho_0 < |z| < 1+\rho_0\big\}$ which assumes only real values for $|z| = 1$.

The monodromy matrices are as follows

$$
\begin{aligned}
M_{[a,b]}(z,\omega,E) &= \prod_{k=b}^{a} A\big(ze(k\omega),\omega,E\big) \\
A(z,\omega,E) &= \begin{bmatrix} V(z)-E & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned} \tag{1.4}
$$

$a,b\in\mathbb{Z}$, $a<b$, $E\in\mathbb{C}$. For $M_{[1,N]}(z,\omega,E)$ we reserve the notation $M_N(z,\omega,E)$. For almost all $z=e(x+iy)\in\mathcal{A}_{\rho_0}$ the limit

$$
\lim_{N\to\infty} N^{-1}\log\big\|M_N(z,\omega,E)\big\| \tag{1.5}
$$

exists; if ω is irrational, then the limit does not depend on x a.s. and it is denoted by $L(y,\omega,E)$. The most important case is $y=0$, and we reserve the notation $L(\omega,E)$ for the Lyapunov exponents $L(0,\omega,E)$. We always assume that the frequency ω satisfies the same Diophantine condition as in [GolSch2], namely

$$
\|n\omega\| \ge \tfrac{c}{n(\log n)^a} \qquad \text{for all } n\ge 1 \tag{1.6}
$$

and some $a>1$. We denote the class of ω satisfying (1.6) by $\mathbb{T}_{c,a}$ and further define

$$
\mathrm{Dioph} := \bigcup_{a>1,c>0} \mathbb{T}_{c,a}
$$

Let $\omega\in\mathrm{Dioph}$. By a theorem of Avron and Simon [AvrSim] the spectrum $\mathrm{spec}(H(x,\omega))$ does not depend on x and we denote it by Σ_ω.

Theorem 1.1. *Assume that $L(\omega,E)>0$ for any $\omega\in(\omega',\omega'')$ and any $E\in(E',E'')$. There exists a set Ω of Hausdorff dimension zero such that for any $\omega\in(\omega',\omega'')\cap\mathrm{Dioph}\backslash\Omega$, the intersection $\Sigma_\omega\bigcap(E',E'')$ is a Cantor set.*

We remark that it follows from this theorem that if $L(\omega_0,E)\ge\gamma>0$ for some $\omega_0\in\mathbb{T}_{c,a}$ and $E_0\in\mathbb{R}$ then there exists $\rho^{(0)}=\rho^{(0)}(V,c,a,\gamma)>0$, and a set Ω of Hausdorff dimension zero such that for any $\omega\in(\omega_0-\rho^{(0)},\omega_0+\rho^{(0)})\cap\mathrm{Dioph}\backslash\Omega$ the spectrum $\mathrm{spec}(H(x,\omega))\bigcap(E_0-\rho^{(0)},E_0+\rho^{(0)})$ is a Cantor set. This is due to the fact that $L(\omega,E)>\gamma/2$ for all $|\omega-\omega_0|<\rho^{(0)}$ (see [GolSch1] or [BouJit]).

Concerning the statement of Theorem 1.1, note that the removal of a set of Hausdorff dimension zero cannot be achieved by a "Fubini"-type argument; rather, it requires some information on the complexity of a suitable cover of sets of bad frequencies ω (relative to finite volume). Throughout this paper we rely heavily on the notion of "complexity" of a set real or complex numbers: if $\mathcal{S}\subset\mathbb{R}$, then

$$
\mathrm{mes}\,(\mathcal{S})<\varepsilon, \quad \mathrm{compl}(\mathcal{S})<K
$$

mean that for some intervals I_k,

$$
\mathcal{S}\subset\bigcup_{k=1}^{K} I_k, \quad \sum_{k=1}^{K}|I_k|<\varepsilon
$$

In all cases considered here, $K\varepsilon\ll 1$ and we will often replace the latter condition by the stronger $\max_k|I_k|<\varepsilon^2$. In the complex case, replace 'interval' by 'disk'. We derive Theorem 1.1 as a simple corollary of our analysis of the gap development in finite volume. The following theorem should be thought of as an (important) representative of the finite volume analysis – the reader will find a more detailed description in later sections. As usual, $[y]$ stands for the entire part of y and $\mathcal{H}^s_\alpha$ is the s-dimensional Hausdorff outer measure of scale α, see e.g. Falconer [Fal]. Finally, we introduce the notation $H_N^{(P)}(x,\omega)$ for the Schrödinger operator on $[-N+1,N]$ with periodic boundary conditions and

$$
\mathcal{S}^{(P)}_{N,\omega} := \bigcup_{x\in\mathbb{T}} \mathrm{spec}(H_N^{(P)}(x,\omega))
$$

Theorem 1.2. *Assume that $L(\omega, E) \geq \gamma > 0$ for any $\omega \in (\omega', \omega'')$ and any $E \in (E', E'')$. Given $c > 0$, and $a > 1$, $0 < s < 1$ there exist positive integers $N_0 = N_0(V, c, a, \gamma, s)$ and $T_0 = T_0(V, c, a, \gamma), A = A(V, c, a, \gamma)$ such that for any $N_1 \geq N_0$ there exists a subset $\Omega_{N_1,s} \subset \mathbb{T}$*

$$\mathcal{H}^s_{\alpha(N_1)}(\Omega_{N_1,s}) \leq 1, \quad \alpha(N_1) = \exp(-(\log\log N_1)^A)$$

such that for all $\omega \in \mathbb{T}_{c,a} \cap (\omega', \omega'')\backslash\Omega_{N_1,s}$ the following statement holds: Set

$$\overline{N}(N_1, 1) := N_1, \quad \overline{N}(N_1, t+1) := [\exp((\overline{N}(N_1, t))^\delta)] \quad \forall\, t \geq 1$$

with some small $0 < \delta = \delta(V, c, a, \gamma) \ll 1$. Then there exists $N \leq \overline{N}(N_1, T_0)$ depending on ω, such that for any interval $I = (\underline{E}, \overline{E})$, $I \subset (E', E'')$ with $|I| > \exp(-(\log N_1)^C)$ there exists a subinterval $I^{(1)} = (\underline{E}^{(1)}, \overline{E}^{(1)}) \subset I$ such that $|I^{(1)}| > \exp(-\overline{N}(N_1, T_0))$ and $\mathcal{S}^{(P)}_{N,\omega} \cap I^{(1)} = \emptyset$.

In the previous theorem, we use the Hausdorff outer measure just for simplicity. In fact, one has the following bounds on the measure and complexity of $\Omega_{N_1,s}$:

$$\Omega_{N_1,s} = \bigcup_{1 \leq t \leq T_0+1} \mathcal{B}_{N_1,t}, \quad \mathrm{mes}\,(\mathcal{B}_{N_1,t}) \leq \mu(t), \quad \mathrm{compl}(\mathcal{B}_{N_1,t}) \leq C(t)$$

where $\mu(t) = \exp(-(\log\log \overline{N}(N_1, t))^A)$, $C(t) = \mu(t)^{-s}$.

It is not clear how to pass from Theorem 1.2 to Theorem 1.1 via the basic definition of the spectrum in ℓ^2 alone. The well-known description of the spectrum via polynomially bounded solutions (via the Schnol–Simon theorem) appears not to be too helpful in this context, either, since it is an existence theorem and thus non-effective. Let us recall in passing that the "proper" spectrum Σ_ω, which is the closure of the set $\{E_j(x, \omega)\}_j$ of the eigenvalues of $H(x, \omega)$, does *not* depend on x, whereas the set $\{E_j(x, \omega)\}_j$ of eigenvalues itself does. The methods developed in [GolSch2], and which are expanded upon here, are centered around the parametrization of the eigenvalues by the phase. Amongst the properties of these parametrizations we single out the crucial *separation* as the most important; it says that the eigenvalues $E_j^{(N)}(x, \omega)$ of $H_{[-N,N]}(x, \omega)$ satisfy

$$|E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega)| > e^{-N^\delta}$$

for any $j \neq k$ provided $E_j^{(N)}(x, \omega) \notin \mathcal{E}_{N,\omega}$ where the latter set is small both with regard to measure and complexity, see Section 7 for more details. In his seminal paper [Sin] on cosine-like potentials, Sinai introduced a (multi-valued) function Λ, defined almost everywhere on $\mathbb{T}$, which allows for the parametrization of the eigenvalues in the *infinite volume* in a co-variant fashion. This means that for almost every $x \in \mathbb{T}$, the set $\{\Lambda(x + j\omega)\}_{j=-\infty}^{\infty}$ is the complete set of eigenvalues of $H(x, \omega)$ and the associated orthonormal basis of eigenfunctions $\{\psi_j(\cdot, x, \omega)\}_{j=-\infty}^{\infty}$ satisfies

$$\psi_j(\cdot, x + \omega, \omega) = \psi_j(\cdot + 1, x, \omega)$$

The following theorem on infinite volume Anderson localization arises as part of our construction of gaps. It is proved via an induction on scales argument with a suitable finite volume localization statement at its core. Amongst other things, it shows that the set of exceptional frequencies ω which need to be removed from all Diophantine ω in order to ensure Anderson localization in the work of Bourgain and the first author, see [BouGol], is of Hausdorff dimension zero. The positive measure statement in the theorem improves on [Bou2] for the same reason. However, our proof of localization is very different technically speaking from the one in [BouGol] and property (4) in the following theorem is new. In essence, this property controls the number of monotonicity intervals of Sinai's function.

Theorem 1.3. *Assume that $L(\omega_0, E) \geq \gamma > 0$ for some $\omega_0 \in \mathbb{T}_{c,a}$ and any $E \in \mathbb{R}$. Then there exist $\rho^{(0)} = \rho^{(0)}(V, c, a, \gamma) > 0$, and a set $\Omega \in \mathbb{T}$ of Hausdorff dimension zero such that for any $\omega \in (\omega_0 - \rho^{(0)}, \omega_0 + \rho^{(0)}) \cap \mathrm{Dioph}\backslash\Omega$ the spectrum satisfies $\mathrm{mes}\,(\Sigma_\omega) > 0$. Furthermore, there exists a set $\mathcal{B}_\omega \subset \mathbb{T}$ of Hausdorff dimension zero, such that for any $x \in \mathbb{T} \setminus \mathcal{B}_\omega$ the following conditions hold:*

(1) *There exists an orthonormal basis* $\{\psi_j(x,\omega,\cdot)\}_{j\ge 1}$ *of eigenfunctions of* $H(x,\omega)$ *in* $\ell^2(\mathbb{Z})$,

$$H(x,\omega)\psi_j(x,\omega,\cdot) = E_j(x,\omega)\psi_j(x,\omega,\cdot)$$

Moreover, each function $\psi_j(x,\omega,\cdot)$ *is exponentially localized and*

$$\lim_{|N|\to\infty}(2|N|)^{-1}\log\left(|\psi_j(x,\omega,N)|^2+|\psi_j(x,\omega,N-1)|^2\right) = -L(\omega,E_j(x,\omega))$$

(2) *The eigenvalues* $E_j(x,\omega)$ *are simple*

(3) *The set* $\mathcal{B}_\omega$ *is invariant under the shifts* $x\mapsto x+m\omega \pmod 1$, $m\in\mathbb{Z}$. *For any* $x\in\mathbb{T}\setminus\mathcal{B}_\omega$, $j\ge 1$, *and* $m\in\mathbb{Z}$, $\psi_j(x,\omega,\cdot+m)$ *is the eigenfunction of* $H(x+m\omega,\omega)$ *with the eigenvalue* $E_j(x,\omega)$

(4) *For each* $E\in\mathbb{R}$ *the set*

$$\mathcal{T}(E) := \{x\in\mathbb{T}\setminus\mathcal{B}_\omega \ :\ \exists j \ \ \textit{so that} \ \ E=E_j(x,\omega)\}$$

is either empty or consists of a union of trajectories $\Gamma(x(E,k))$, $x(E,k)\in\mathbb{T}\setminus\mathcal{B}_\omega$, $1\le k\le k(E)$ *where* $k(E)\le C(V)<\infty$ *and*

$$\Gamma(x) := x+\omega\mathbb{Z} \pmod 1$$

If V *is a trigonometric polynomial of degree*[1] k_0, *then* $C(V)\le 2k_0$.

We feel that the methods of this paper, combined with some "soft" measure theoretic consideration, should allow for the construction of a true Sinai's function. I.e., we claim that there exists a function $\Lambda : \mathbb{T}\to\mathbb{R}$, defined up to a set of Hausdorff dimension zero and with at most $C(V)$ monotonicity intervals where $C(V)$ is as in part (4) above such that

$$E_j(x,\omega) = \Lambda(x+j\omega) \qquad \forall j\in\mathbb{Z},\ \forall\, x\in\mathbb{T}\setminus\Omega$$

where Ω is of Hausdorff dimension zero. However, we have chosen not to pursue this issue here.
In Section 13 we derive a detailed finite volume version of Theorem 1.3. Amongst other things, this derivation gives an effective quantitative description of the spectrum of the problem (1.1) on the whole lattice $\mathbb{Z}$ in terms of the spectrum on finite volume and also allows for a simple transition from Theorem 1.2 to Theorem 1.1. The co-variant parametrization of the eigenvalues and eigenfunctions via the phases is based on the description of the exponentially localized eigenfunctions on the interval $[-\overline{N},\overline{N}]$ by means the eigenfunction on the interval $[-N,N]$ with $N\le\overline{N}$, combined with the aforementioned quantitative repulsion property of the Dirichlet eigenvalues on a finite volume. We discuss these results in Sections 5–7. As already mentioned before, we produce gaps (on finite volume) from resonances of the previous scale. This requires restricting the graphs of the eigenfunctions to segments which have a controlled slope (in the sense of a favorable lower bound). Thus, in Section 10 we introduce I–segments $\{E_j^{(N)}(x,\omega),\underline{x},\bar{x}\}$ of the graphs of the eigenfunctions on a finite volume $[-N,N]$ which have slopes bounded below (in absolute value) by e^{-N^δ}. Amongst those we single out *regular* I-segments which have the property that the eigenfunctions with eigenvalues $E_j^{(N)}(x,\omega)$ are supported away from the boundary of $[-N,N]$ for all $\underline{x}<x<\bar{x}$. This is needed in order to assure that crossing I-segments do indeed form a resonance at a larger scale as in the figures above. For that we use the spectrum and the eigenfunctions of the Schrödinger operator on a finite interval with periodic (respectively, antiperiodic) boundary conditions. The key analytical tool for the study of periodic boundary conditions consists of a large deviation estimate for the trace of the propagator matrix M_N which we derive in Section 3. We now describe the strategy behind the proof of Theorem 1.2 in more detail. One first shows that segments $E_1(x),E_2(x)$ as in the matrix (1.3) exist. Invoking the estimates for the separation of the Dirichlet eigenvalues and the zeros of the Dirichlet determinants established in [GolSch2], one next shows that the resonance defined by E_1,E_2 leads to two new eigenvalues $E^+(x),E^-(x)$ of the "next scale", see Figure 2. We call the interval

$$\left(\max_{x\in J}E^-(x),\min_{x\in J}E^+(x)\right)$$

[1] This means that $V(x)=\sum_{k=-k_0}^{k_0}a_k e(kx)$ with $a_{-k}=\overline{a_k}$.

a *pre-gap* at scale $\overline{N}$. The interval J here is the common domain of E_1 and E_2. At this point one faces the obstruction of a so-called *triple resonance*. Recall that the resonance defined by (1.2) is called a *double resonance* if, with $B \gg A$ from (1.2),

$$\Big|E_{j_1}^{(N)}(x,\omega) - E_{j_3}^{(N)}(x+m'\omega,\omega)\Big| > (m')^{-B} \tag{1.7}$$

for any pair $(j_3, m') \neq (j_2, m)$, with $m \ll m' \le \overline{N}$, where $\overline{N} \asymp \exp\big(N^\delta\big)$ is the "next scale". Otherwise it is called a triple (or higher order) resonance.

As mentioned above the triple resonance obstruction already appears in Sinai's perturbative method [Sin], see also Bourgain's paper on almost Mathieu [Bou1]. In fact, by the choice of a cosine-like potential and for large $|\lambda|$ this type of resonance is excluded in [Sin] and [Bou1]. For general potentials, it was shown by J. Chan [Cha] that if

$$\big|\partial_x E_j^{(N)}(x,\omega)\big| + \big|\partial_{xx} E_j^{(N)}(x,\omega)\big| > \lambda(N,\omega) > 0$$

with a suitable function $\lambda(N,\omega)$, then triple resonances do not occur for most ω. In Section 14 we follow a similar approach in the case where the graphs $E_j^{(N)}(x,\omega)$ have controlled slopes. Moreover, for the case of analytic potentials, one can show that the triple (or higher) resonance obstruction can occur only for a set of frequencies of Hausdorff dimension zero.

In order to run an "induction on scales" argument that shows how pre-gaps in finite volume eventually lead to gaps in infinite volume, we invoke the mechanism of counting complex zeros of the characteristic determinants of $H_N(x,\omega)$ as developed in [GolSch2]. Using complexified notations, the characteristic determinants are as follows:

$$\begin{aligned} f_N(z,\omega,E) &= \det\big(H_N(z,\omega)-E\big) \\ &= \begin{vmatrix} V\big(ze(\omega)\big)-E & -1 & 0 & \cdots & \cdots & 0 \\ -1 & V\big(ze(2\omega)\big)-E & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & -1 \\ 0 & \dots\dots\dots\dots\dots & 0 & -1 & & V\big(ze(N\omega)\big)-E \end{vmatrix} \end{aligned} \tag{1.8}$$

where

$$f_{[a,b]}(z,\omega,E) = f_{b-a+1}\big(ze(a\omega),\omega,E\big) \tag{1.9}$$

For future reference, we remark that for any interval $\Lambda \subset \mathbb{Z}$, $H_\Lambda(z,\omega)$ denotes the matrix obtained from $H(z,\omega)$ by restriction to Λ with Dirichlet boundary conditions; in addition, we let $f_\Lambda := \det(H_\Lambda - E)$. As already remarked above, we write H_N and f_N for $H_{[1,N]}$ and $f_{[1,N]}$, respectively (although occasionally, the same notation will also be used relative to the interval $[-N,N]$). It is well–know that these functions are closely related to the monodromy (or propagator) matrices (1.4). In fact,

$$M_N(z,\omega,E) = \begin{bmatrix} f_N(z,\omega,E) & -f_{N-1}\big(ze(\omega),\omega,E\big) \\ f_{N-1}(z,\omega,E) & -f_{N-2}\big(ze(\omega),\omega,E\big) \end{bmatrix} \tag{1.10}$$

By means of this relation, large deviation estimates and an avalanche principle expansion for the function $\log\big|f_N(z,\omega,E)\big|$ were developed in [GolSch2]. In Section 2 we recall the statements of these results and prove some corollaries. These corollaries, combined with a suitable version of the Jensen formula (see (e) in Section 2) enable one to locate and count the zeros of $f_N(\cdot,\omega,E)$ in the annulus $\mathcal{A}_{\rho_0}$ and its subdomains. In particular, this technique allows one to claim that if

$$E \in \big(\max_x E^-(x) + \exp\big(-\overline{N}^{1/2}\big), \min_x E^+(x) - \exp\big(-\overline{N}^{1/2}\big)\big),$$

where $\big(\max E^-(x), \min E^+(x)\big)$ is a pre-gap at scale $\overline{N}$, then $f_{\overline{N}}(\cdot,\omega,E)$ has two complex zeros $\zeta_\ell = e(x_\ell + iy_\ell)$, with $\exp\big(-N^\delta\big) > |y_\ell| > \exp\big(-\overline{N}^\delta\big)$, $\ell = 1, 2$. This is due to the absence of triple resonances

and the stability of the number of zeros of $f_{\overline{N}}(\cdot,\omega,E)$ under small perturbations of E. The most effective form of the last property consists of the Weierstrass preparation theorem for $f_N(\cdot,\omega,E)$, which is described in part (g) of Section 2. To complete the description of the formation of a gap from a pre-gap we use the translations of the segments $\{E^{(N)}_{j_1}(x), \underline{x}, \bar{x}\}$ under the shifts $x \mapsto x + k\omega$. Using the localization property of eigenfunctions on a finite interval (see Section 6), we show that if a double resonance (1.2) occurs then the same is true for a sequence of segments which are "almost" identical with the shifts $E_{j_1}(x+k\omega), E_{j_2}(x+k\omega)$, $1 \le k \le \overline{N}\big(1-0(1)\big)$. This method is explained in great detail in Sections 11-12. In particular, the possible locations of the "center of localization" of an eigenfunction plays an important role with the "bad case" being when this center is too close to the boundary of the finite volume interval. Due to this method we obtain a whole sequence of complex zeros $\zeta_{k,\ell} \cong e\big(x_\ell + k\omega + iy_\ell\big)$ of $f_N(\cdot,\omega,E)$. So, the numbers

$$\mathcal{M}_N(E) = N^{-1}\#\big\{z : 1-\rho_N < |z| < 1+\rho_N, f_N(z,\omega,E) = 0\big\}$$

$\rho_N = \exp\big(-N^\delta\big)$ decrease at least by $2-o(1)$ by going from scale N to scale $\overline{N}$, provided E is in the pre-gap. After a finite number of inductive steps one can locate a gap and complete the proof of Theorem 1.2. Needless to say, the zero counting mechanism for the Dirichlet determinants from [GolSch2] plays a crucial role here. The relevant sections in this regard are 4 and 9, where the applications of the Jensen formula and the avalanche principle expansions, respectively, are presented. For a summary of [GolSch2] see [GolSch4], and for a heuristic discussion of gaps in the context of this paper see [GolSch3].

2. A review of the basic tools

In this section we give a sketch of the main ingredients of the method developed in [GolSch2]. We of course do not reproduce all the material from that paper in full detail, and refer the reader for most proofs to [GolSch2]. We start our discussion with the classical Cartan estimate for analytic functions.

(a) Cartan Estimate

Definition 2.1. *Let $H \gg 1$. For an arbitrary subset $\mathcal{B} \subset \mathcal{D}(z_0,1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \mathrm{Car}_1(H,K)$ if $\mathcal{B} \subset \bigcup_{j=1}^{j_0} \mathcal{D}(z_j,r_j)$ with $j_0 \le K$, and*

$$\sum_j r_j < e^{-H}\ . \tag{2.1}$$

If d is a positive integer greater than one and $\mathcal{B} \subset \prod_{i=1}^{d} \mathcal{D}(z_{i,0},1) \subset \mathbb{C}^d$ then we define inductively that $\mathcal{B} \in \mathrm{Car}_d(H,K)$ if for any $1 \le j \le d$ there exists $\mathcal{B}_j \subset \mathcal{D}(z_{j,0},1) \subset \mathbb{C}, \mathcal{B}_j \in \mathrm{Car}_1(H,K)$ so that $\mathcal{B}^{(j)}_z \in \mathrm{Car}_{d-1}(H,K)$ for any $z \in \mathbb{C}\setminus\mathcal{B}_j$, here $\mathcal{B}^{(j)}_z = \big\{(z_1,\dots,z_d) \in \mathcal{B} : z_j = z\big\}$.

Remark 2.2. *(a) This definition is consistent with the notation of Theorem 4 in Levin's book* [Lev], *p. 79.*
(b) It is important in the definition of $\mathrm{Car}_d(H,K)$ for $d > 1$ that we control both the measure and the complexity K of each slice $\mathcal{B}^{(j)}_z$, $1 \le j \le d$.

The following lemma is a straightforward consequence of this definition.

Lemma 2.3.

1. *Let $\mathcal{B}_j \in \mathrm{Car}_d(H,K)$, $\mathcal{B}_j \subset \prod_{j=1}^{d} \mathcal{D}(z_{j,0},1)$, $j = 1,2,\dots,T$. Then $\mathcal{B} = \bigcup_j \mathcal{B}_j \in \mathrm{Car}_d\big(H-\log T, TK\big)$.*
2. *Let*

$$\mathcal{B} \in \mathrm{Car}_d(H,K), \quad \mathcal{B} \subset \prod_{j=1}^{d} \mathcal{D}\big(z_{j,0},1\big)$$

Then there exists

$$\mathcal{B}' \in \mathrm{Car}_{d-1}(H,K), \quad \mathcal{B}' \subset \prod_{j=2}^{d} \mathcal{D}\big(z_{j,0},1\big)$$

such that $\mathcal{B}_{(w_2,\dots,w_d)} \in \mathrm{Car}_1(H,K)$*, for any* $(w_2,\dots,w_d)\in\mathcal{B}'$*.*

Next, we generalize the usual Cartan estimate to several variables.

Lemma 2.4. *Let* $\varphi(z_1,\dots,z_d)$ *be an analytic function defined in a polydisk* $\mathcal{P}=\prod_{j=1}^{d}\mathcal{D}(z_{j,0},1)$*,* $z_{j,0}\in\mathbb{C}$*. Let* $M \ge \sup_{\underline{z}\in\mathcal{P}}\log|\varphi(\underline{z})|$*,* $m\le \log\big|\varphi(\underline{z}_0)\big|$*,* $\underline{z}_0=(z_{1,0},\dots,z_{d,0})$*. Given* $H\gg 1$ *there exists a set* $\mathcal{B}\subset\mathcal{P}$*,* $\mathcal{B}\in\mathrm{Car}_d\big(H^{1/d},K\big)$*,* $K=C_dH(M-m)$*, such that*

$$\log\big|\varphi(z)\big| > M - C_dH(M-m) \tag{2.2}$$

for any $z\in\prod_{j=1}^{d}\mathcal{D}(z_{j,0},1/6)\setminus\mathcal{B}$*.*

Proof. The proof goes by induction over d. For $d=1$ the assertion is Cartan's estimate for analytic functions. Indeed, Theorem 4 on page 79 in [Lev] applied to $f(z)=e^{-m}\varphi(z)$ yields that

$$\log\big|\varphi(z)\big| > m - CH(M-m) = M-(CH+1)(M-m)$$

holds outside of a collection of disks $\{\mathcal{D}(a_k,r_k)\}_{k=1}^{K}$ with $\sum_{k=1}^{K} r_k \lesssim \exp(-H)$. Increasing the constant C leads to (2.2). Moreover, $K/5$ cannot exceed the number of zeros of the function $\varphi(z)$ in the disk $\mathcal{D}(z_{1,0},1)$ counted with multiplicity, which is in turn estimated by Jensen's formula, as $\lesssim M-m$, see the following section. Although this bound on K is not explicitly stated in Theorem 4 in [Lev], it can be deduced from the proofs of Theorems 3 and 4 in [Lev]. Indeed, one can assume that each of the disks $\mathcal{D}(a_k,r_k)$ contains a zero of φ, and it is shown in the proof of Theorem 3 in [Lev] that no point is contained in more than five of these disks. Hence we have proved the $d=1$ case with a bad set $\mathcal{B}\in\mathrm{Car}_1(H,C(M-m))$, which is slightly better than stated above (the H dependence of K appears if $d>1$ and we will ignore some slight improvements that are possible to the statement of the lemma due to this issue).

In the general case take $1\le j\le d$ and consider $\psi(z)=\varphi\big(z_{1,0},\dots,z_{j-1,0},z,z_{j+1,0},\dots,z_{d,0}\big)$. Due to the $d=1$ case there exists $\mathcal{B}^{(j)}\in\mathrm{Car}_1\big(H^{1/d},C_1(M-m)\big)$, such that

$$\log\big|\psi(z)\big| > M - C_1H^{1/d}(M-m)$$

for any $z\in\mathcal{D}\big(z_{j,0},1/6\big)\setminus\mathcal{B}^{(j)}$. Take arbitrary $z_{j,1}\in\mathcal{D}\big(z_{j,0},1/6\big)\setminus\mathcal{B}^{(j)}$ and consider the function

$$\chi\big(z_1,z_2,\dots,z_{j-1},z_{j+1},\dots,z_d\big)=\varphi\big(z_1,\dots,z_{j-1},z_{j,1},z_{j+1},\dots,z_d\big)$$

in the polydisk $\mathcal{P}':=\prod_{i\ne j}\mathcal{D}\big(z_{i,0},1\big)$. Then

$$\sup_{\mathcal{P}'}\log\big|\chi(z_1,\dots,z_{j-1},z_{j+1},\dots,z_d)\big| \le M,$$
$$\log\big|\chi(z_{1,0},\dots,z_{j-1,0},z_{j+1,0},\dots,z_{d,0})\big| > M - CH^{1/d}(M-m).$$

Thus χ satisfies the conditions of the lemma with the same M and with m replaced with

$$M-CH^{1/d}(M-m).$$

We now apply the inductive assumption for $d-1$ and with H replaced with $H^{\frac{d-1}{d}}$ to finish the proof. $\square$

Later we will need the following general assertion which is a combination of the Cartan-type estimate of the previous lemma and Jensen's formula on the zeros of analytic functions, see (e) of the present section.

Lemma 2.5. *Fix some $\underline{w}_0 = (w_{1,0}, w_{2,0}, \dots, w_{d,0}) \in \mathbb{C}^d$ and suppose that $f(\underline{w})$ is an analytic function in $\mathcal{P} = \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, 1)$. Assume that $M \geq \sup_{\underline{w}\in\mathcal{P}} \log|f(\underline{w})|$, and let $m \leq \log|f(\underline{w}_1)|$ for some $\underline{w}_1 = (w_{1,1}, w_{2,1}, \dots, w_{d,1}) \in \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, 1/2)$. Given $H \gg 1$ there exists $\mathcal{B}'_H \subset \mathcal{P}' = \prod_{j=2}^{d} \mathcal{D}(w_{j,0}, 3/4)$, $\mathcal{B}'_H \in \mathrm{Car}_{d-1}\big(H^{1/d}, K\big)$, $K = CH(M-m)$ such that for any $\underline{w}' = (w_2, \dots, w_d) \in \mathcal{P}' \setminus \mathcal{B}'_H$ the following holds: if*

$$\log|f(\tilde{w}_1, \underline{w}')| < M - C_d H(M-m) \quad \text{for some} \quad \tilde{w}_1 \in \mathcal{D}(w_{1,0}, 1/2),$$

then there exists $\hat{w}_1$ with $|\hat{w}_1 - \tilde{w}_1| \lesssim e^{-H^{\frac{1}{d}}}$ such that $f(\hat{w}_1, \underline{w}') = 0$.

Proof. Due to Lemma 2.4, there exists $\mathcal{B}_H \subset \mathcal{P}$, $\mathcal{B}_H \in \mathrm{Car}_d\big(H^{1/d}, K\big)$, $K = C_d H(M-m)$ such that for any $\underline{w} \in \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, 3/4) \setminus \mathcal{B}_H$ one has

$$\log\big|f(\underline{w})\big| > M - C_d H(M-m)\ . \tag{2.3}$$

By Lemma 2.3, part (2), there exists $\mathcal{B}'_H \subset \prod_{j=2}^{d} \mathcal{D}\big(w_{j,0}, 1\big)$, $\mathcal{B}'_H \in \mathrm{Car}_{d-1}(H^{\frac{1}{d}}, K)$ such that $\big(\mathcal{B}_H\big)_{\underline{w}'} \in \mathrm{Car}_1(H^{\frac{1}{d}}, K)$ for any $\underline{w}' = (w_2, \dots, w_d) \in \mathcal{B}'_H$. Here $(\mathcal{B})_{\underline{w}'}$ stands for the $\underline{w}'$–section of $\mathcal{B}$. Assume

$$\log\big|f(\tilde{w}_1, \underline{w}')\big| < M - C_d H(M-m)$$

for some $\tilde{w}_1 \in \mathcal{D}(w_{1,0}, 1/2)$, and $\underline{w}' \in \mathcal{P}' \setminus \mathcal{B}'_H$. Since $\big(\mathcal{B}_H\big)_{\underline{w}'} \in \mathrm{Car}_1(H^{\frac{1}{d}}, K)$ there exists $r \lesssim \exp\big(-H^{1/d}\big)$ such that

$$\big\{z : |z - \tilde{w}_1| = r\big\} \cap \big(\mathcal{B}_H\big)_{\underline{w}'} = \emptyset\ .$$

Then in view of (2.3),

$$\log\big|f(z, \underline{w}')\big| > M - C_d H(M-m)$$

for any $|z - \tilde{w}_1| = r$. It follows from the maximum principle that $f(\cdot, \underline{w}')$ has at least one zero in the disk $\mathcal{D}(\tilde{w}_1, r)$, as claimed. ∎

(b) Large deviation theorem for the monodromies and their entries

Let $M_n(z, \omega, E)$ be the monodromies defined as in (1.4). The entries of $M_n(z, \omega, E)$ are the determinants $f_{[1+a, N-b]}(z, \omega, E)$, $a, b \in \{0, 1\}$, see (1.8), (1.9). Let

$$L(y, \omega, E) = \lim_{N\to\infty} N^{-1} \int \log\|M_N(e(x+iy), \omega, E)\|\, dx$$

be the Lyapunov exponent. We shall assume throughout this paper that the Lyapunov exponents are bounded away from zero; the positive lower bounded on the Lyapunov exponent will typically be denoted by γ. We shall also adhere to the following convention regarding constants for the remainder of the paper:

Definition 2.6. *Constants appearing in the paper will be denoted by A, B, C as well as A_j, B_j, C_j, $j \geq 0$. As a rule, they will be allowed to depend on ω, γ, V, E. The dependence on V will only be exclusively through $\rho_0 > 0$ and $\|V\|_{L^\infty(\mathcal{A}_{\rho_0})}$ where V is analytic on the annulus $\mathcal{A}_{\rho_0}$. Moreover, the dependence on ω will be only through a, c where $\omega \in \mathbb{T}_{c,a}$. Finally, constants depending on E will be uniform for E ranging over bounded sets. For any positive numbers a, b we let $a \lesssim b$ denote $a \leq Cb$ and $a \ll b$ denote $a \leq C^{-1}b$. Finally, $a \asymp b$ stands for $a \lesssim b$ and $b \lesssim a$.*

We now state the large deviation estimates (LDEs) which are fundamental to the arguments of this paper. It will be assumed tacitly that V is analytic on $\mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$ and Definition 2.6 will be in force. However, we shall for now *not assume* that V is real-valued.

Proposition 2.7. *Assume that $L(y,\omega,E) \ge \gamma > 0$ for some $y \in (-\rho_0/10, \rho_0/10)$, $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{C}$. Then for any $N \ge 2$,*

$$\operatorname{mes}\{x \in \mathbb{T} \,:\, \big|\log\|M_N(e(x+iy),\omega,E)\| - NL(y,\omega,E)\big| > H\} \le C\exp\big(-H/(\log N)^{C_0}\big) \tag{2.4}$$

$$\operatorname{mes}\{x \in \mathbb{T} \,:\, \big|\log|f_N(e(x+iy),\omega,E)| - NL(y,\omega,E)\big| > H\} \le C\exp\big(-H/(\log N)^{C_0}\big) \tag{2.5}$$

for all $H > (\log N)^{C_0}$.

We remark that it makes no difference here whether we write NL or NL_N. This is due to the estimate

$$0 \le L_N(\omega,E) - L(\omega,E) \le \frac{C}{N} \quad \forall\, N \ge 1$$

from [GolSch1]. The bound (2.4) for the monodromies (in an even sharper form) is in [GolSch1]. In [GolSch2] it is shown how to pass from (2.4) to (2.5), see Sections 2, 3 of that paper.

(c) The avalanche principle expansion for the Dirichlet determinants

Another basic tool in this paper is the following avalanche principle, see [GolSch1] and [GolSch2].

Proposition 2.8. *Let $A_1, \dots, A_n$ be a sequence of 2×2–matrices whose determinants satisfy*

$$\max_{1\le j\le n} |\det A_j| \le 1. \tag{2.6}$$

Suppose that

$$\min_{1\le j\le n} \|A_j\| \ge \mu > n \quad \text{and} \tag{2.7}$$

$$\max_{1\le j<n} [\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|] < \frac{1}{2}\log\mu. \tag{2.8}$$

Then

$$\Big|\log\|A_n\cdot\ldots\cdot A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\Big| < C\frac{n}{\mu} \tag{2.9}$$

with some absolute constant C.

Combining this with the large deviation theorems from above yields the following expansion for the determinants. Let $f_N(z,\omega,E)$ be the determinants defined as in (1.8), and let $L(\omega,E)$ be the Lyapunov exponent as above but with $y=0$. As before, V is analytic on $\mathcal{A}_{\rho_0}$. For convenience we will assume that V is real-valued. As mentioned above, any constant depending on V depends only on ρ_0 and $\|V\|_{L^\infty(\mathcal{A}_{\rho_0})}$.

Corollary 2.9. *Assume that $L(\omega,E) \ge \gamma > 0$ for some $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{C}$. There exists $N_0 = N_0(V,\omega,\gamma,E)$, $\rho^{(0)} = \rho^{(0)}(V,\omega,\gamma,E) > 0$ such that for any $N \ge N_0(V,\omega,\gamma,E)$ and any integers $\ell_1,\dots,\ell_n$, $(\log N)^{C_0} < \ell_j < cN$ (where $C_0 = C_0(a)$ is a large constant), $\sum_j \ell_j = N$ the following expansion is valid:*

$$\log\big|f_N(e(x+iy),\omega,E)\big| = \sum_{j=1}^{n-1}\log\|A_{j+1}(z)\,A_j(z)\| - \sum_{j=2}^{n-1}\log\|A_j(z)\| + O\big(\exp(-\underline{\ell}^{1/2})\big) \tag{2.10}$$

for any $z = e(x+iy) \in \mathcal{A}_{\rho_0} \setminus \mathcal{B}_{N,\omega,E}$, where

$$\mathcal{B}_{N,\omega,E} = \bigcup_{k=1}^{k_0} \mathcal{D}\big(\zeta_k, \exp(-\underline{\ell}^{1/2})\big)\ , \quad \underline{\ell} = \min_j \ell_j, \quad k_0 \lesssim N,$$
$$A_m(z) = M_{\ell_m}\big(ze(s_m\omega),\omega,E\big), \qquad m = 2,\dots,n-1$$
$$A_1(z) = M_{\ell_1}(z,\omega,E)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$
$$A_n(z) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} M_{\ell_n}\big(ze(s_n\omega),\omega,E\big)$$

and with $s_m = \sum_{j<m} \ell_j$.

A detailed derivation of this theorem can be found in Sections 2 and 3 of [GolSch2].

(e) Uniform upper estimates on the norms of monodromy matrices

The proof of the uniform upper estimate is based on an application of the avalanche principle expansion in combination with the following useful general property of averages of subharmonic functions.

Lemma 2.10. *Let* $1 > \rho > 0$ *and suppose* u *is subharmonic on* $\mathcal{A}_\rho$ *such that* $\sup_{z\in\mathcal{A}_\rho} u(z) \le 1$ *and* $\int_{\mathbb{T}} u(e(x))\,dx \ge 0$. *Then for any* r_1, r_2 *so that* $1 - \frac{\rho}{2} < r_1, r_2 < 1 + \frac{\rho}{2}$ *one has*

$$|\langle u(r_1 e(\cdot))\rangle - \langle u(r_2 e(\cdot))\rangle| \le C_\rho\, |r_1 - r_2|,$$

here $\langle v(\cdot)\rangle = \int_0^1 v(\xi)\,d\xi$.

For the proof see Lemma 4.1 in [GolSch2]. This assertion immediately implies the following corollary regarding the continuity of L_N in y.

Corollary 2.11. *Let* $L_N(y,\omega,E)$ *and* $L(y,\omega,E)$ *be defined as above. Then with some constant* $\rho > 0$ *that is determined by the potential,*

$$|L_N(y_1,\omega,E) - L_N(y_2,\omega,E)| \le C|y_1 - y_2| \quad \text{for all} \quad |y_1|, |y_2| < \rho$$

uniformly in N. *In particular, the same bound holds for* L *instead of* L_N *so that*

$$\inf_E L(\omega,E) > \gamma > 0$$

implies that

$$\inf_{E, |y| \ll \gamma} L(y,\omega,E) > \frac{\gamma}{2}.$$

The following result improves on the uniform upper bound on the monodromy matrices from [BouGol] and [GolSch1]. The $(\log N)^A$ error here (rather than N^σ, say, as in [BouGol] and [GolSch1]) is crucial for the study of the distribution of the zeros of the determinants and eigenvalues, see Proposition 4.3 in [GolSch2]. We remind the reader of our convention regarding constants, see Definition 2.6.

Proposition 2.12. *Assume* $L(\omega,E) \ge \gamma > 0$, $\omega \in \mathbb{T}_{c,a}$. *Then*

$$\sup_{x\in\mathbb{T}} \log \|M_N(x,\omega,E)\| \le N L_N(\omega,E) + C(\log N)^{C_0} \ ,$$

for all $N \ge 2$.

We now list some applications of this upper bound. See Section 4 of [GolSch2].

Corollary 2.13. *Fix* $\omega_1 \in \mathbb{T}_{c,a}$ *and* $E_1 \in \mathbb{C}$, $|y| < \rho_0$. *Assume that* $L(y,\omega_1,E_1) \ge \gamma > 0$. *Then*

$$\begin{aligned} &\sup\bigl\{\log \bigl\|M_N\bigl(e(x+iy),\omega,E\bigr)\bigr\| : |E - E_1| + |\omega - \omega_1| < N^{-C},\, x \in \mathbb{T}\bigr\} \\ &\qquad \le N L_N(y,\omega_1,E_1) + C(\log N)^{C_0} \end{aligned}$$

for all $N \ge 2$.

The importance here lies with the large size of the perturbations: a crude argument would only allow for perturbations of size e^{-CN}. To achieve the much larger size N^{-C} one needs to invoke the avalanche principle with smaller factors of size $\ell \asymp \log N$ which is allowed by the sharp LDE (on scale ℓ) from [GolSch1].

Corollary 2.14. *Fix* $\omega_1 \in \mathbb{T}_{c,a}$ *and* $E_1 \in \mathbb{C}$, $|y| < \rho_0$. *Assume that* $L(y,\omega_1,E_1) \ge \gamma > 0$. *Let* ∂ *denote any of the partial derivatives* $\partial_x, \partial_y, \partial_E$ *or* ∂_ω. *Then*

$$\begin{aligned} &\sup\bigl\{\log \bigl\|\partial M_N\bigl(e(x+iy),\omega,E\bigr)\bigr\| : |E - E_1| + |\omega - \omega_1| < N^{-C}, x \in \mathbb{T}\bigr\} \\ &\qquad \le N L_N(y,\omega_1,E_1) + C(\log N)^{C_0} \end{aligned}$$

for all $N \ge 2$. *Here* $C_1 = C_1(a)$ *and* $C = C(V,\rho_0,a,c,\gamma,E_1)$.

Proof. Clearly, for all x, y, ω, E,

$$\begin{aligned}&\partial M_N\big(e(x+iy),\omega,E\big)\\&=\sum_{n=1}^{N} M_{N-n}\big(e(x+n\omega+iy),\omega,E\big)\,\partial\begin{bmatrix}V(e(x+n\omega+iy))-E & -1\\ 1 & 0\end{bmatrix} M_{n-1}\big(e(x+iy),\omega,E\big)\end{aligned}$$

Since $|E-E_1|+|\omega-\omega_1|<N^{-C}$, the statement now follows from Corollary 2.13, Corollary 2.11, as well as the rate of convergence estimate

$$0\le L_N(\omega,E)-L(\omega,E)\le\frac{C}{N},\quad \forall\, N\ge 2$$

from [GolSch1]. ☐

The previous bound on the derivatives implies the following bound on differences of propagator matrices.

Corollary 2.15. *Under the assumptions of the previous corollary,*

$$\begin{aligned}&\big\|M_N\big(e(x+iy),\omega,E\big)-M_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big\|\\&\le\big(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|\big)\cdot\exp\big(NL_N\big(y_1,\omega_1,E_1\big)+C(\log N)^{C_0}\big)\end{aligned}$$

provided $|E-E_1|+|\omega-\omega_1|+|x-x_1|<N^{-C}$, $|y_1|<\rho_0/2$, $|y-y_1|<N^{-1}$. *In particular,*

$$\begin{aligned}\Big|\log\frac{\big|f_N\big(e(x+iy),\omega,E\big)\big|}{\big|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\Big|&\le\big(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|\big)\\&\qquad\frac{\exp\big(NL(y_1,\omega_1,E_1)+C(\log N)^{C_0}\big)}{\big|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\ ,\end{aligned}\tag{2.11}$$

for all $N\ge 2$ provided the right-hand side of (2.11) *is less than* $1/2$.

Proof. For (2.11) estimate

$$\begin{aligned}&|f_N\big(e(x+iy),\omega,E\big)-f_N\big(e(x_1+iy_1),\omega_1,E_1\big)|\\&\lesssim(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|)\sup|df_N(e(x'+iy'),\omega',E')|\end{aligned}\tag{2.12}$$

where the supremum is taken over all x', y', ω', E' on the line joining (x,y,ω,E) to (x_1,y_1,ω_1,E_1) and d stands for the derivative in all variables. By Corollary 2.14 we can bound

$$\sup|df_N(e(x'+iy'),\omega',E')|\lesssim\exp(NL(y_1,\omega_1,E_1)+C(\log N)^{C_0})$$

Dividing (2.12) by $f_N\big(e(x_1+iy_1),\omega_1,E_1\big)$ therefore yields

$$\begin{aligned}&\Big|\frac{\big|f_N\big(e(x+iy),\omega,E\big)\big|}{\big|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}-1\Big|\\&\lesssim\big(|E-E_1|+|\omega-\omega_1|+|x-x_1|+|y-y_1|\big)\frac{\exp\big(NL(y_1,\omega_1,E_1)+C(\log N)^{C_0}\big)}{\big|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\end{aligned}$$

By assumption, the right-hand side here is $<\frac{1}{2}$. Hence, (2.11) follows by taking logarithms. ☐

A particular instance of this bound is the following one.

Corollary 2.16. *Using the notation of the previous corollary one has*

$$\Big|\log\frac{\big\|M_N\big(e(x+iy),\omega,E\big)\big\|}{\big\|M_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big\|}\Big|<C\exp\big(-(\log N)^{C_0}\big)\tag{2.13}$$

$$\Big|\log\frac{\big|f_N\big(e(x+iy),\omega,E\big)\big|}{\big|f_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\Big|<C\exp\big(-(\log N)^{C_0}\big)\tag{2.14}$$

for any $|E - E_1| + |\omega - \omega_1| + |x - x_1| + |y - y_1| < \exp\bigl(-(\log N)^{4C_0}\bigr)$, $x_1 \in \mathcal{A}_{\rho_0/2} \setminus \mathcal{B}_{y_1,\omega_1,E_1}$, *where* $\mathrm{mes}\,(\mathcal{B}_{y_1,\omega_1,E_1}) < \exp\bigl(-(\log N)^{C_0}\bigr)$, $\mathrm{compl}(\mathcal{B}_{y_1,\omega_1,E_1}) \le CN$. *In particular,*

$$|L(y,\omega,E) - L(y_1,\omega_1,E_1)| \le C\exp(-(\log N)^{C_0}) \tag{2.15}$$

provided $|E - E_1| + |\omega - \omega_1| + |y - y_1| < \exp\bigl(-(\log N)^{4C_0}\bigr)$.

An important application of the uniform upper bounds is the following analogue of Wegner's estimate from the random case. We provide the proof here just to demonstrate how the previous corollaries can be applied.

Lemma 2.17. *Let V be analytic and real-valued on $\mathbb{T}$ as in the previous result. Suppose $\omega \in \mathbb{T}_{c,a}$. Then for any $E \in \mathbb{R}$, $H \ge (\log N)^{C_0}$ one has*

$$\mathrm{mes}\,\bigl\{x \in \mathbb{T} \,:\, \mathrm{dist}\bigl(\mathrm{spec}(H_N(x,\omega)), E\bigr) < \exp(-H)\bigr\} \lesssim \exp\bigl(-H/(\log N)^{C_0}\bigr) \tag{2.16}$$

for all $N \ge 2$. Moreover, the set on the left-hand side is contained in the union of $\lesssim N$ intervals each of which does not exceed the bound stated in (2.16) *in measure.*

Proof. By Cramer's rule

$$\bigl|\bigl(H_N(x,\omega) - E\bigr)^{-1}(k,m)\bigr| = \frac{\bigl|f_{[1,k]}\bigl(e(x),\omega,E\bigr)\bigr|\,\bigl|f_{[m+1,N]}\bigl(e(x),\omega,E\bigr)\bigr|}{\bigl|f_N\bigl(e(x),\omega,E\bigr)\bigr|} \tag{2.17}$$

By Proposition 2.12

$$\log\bigl|f_{[1,k]}\bigl(e(x),\omega,E\bigr)\bigr| + \log\bigl|f_{[m+1,N]}\bigl(e(x),\omega,E\bigr)\bigr| \le NL(\omega,E) + C(\log N)^{C_0}$$

for any $x \in \mathbb{T}$. Therefore,

$$\bigl\|\bigl(H_N(x,\omega) - E\bigr)^{-1}\bigr\| \le N^2\,\frac{\exp\bigl(NL(\omega,E) + C(\log N)^{C_0}\bigr)}{\bigl|f_N\bigl(e(x),\omega,E\bigr)\bigr|} \tag{2.18}$$

for any $x \in \mathbb{T}$. Since

$$\mathrm{dist}\bigl(\mathrm{spec}\bigl(H_N(x,\omega),E\bigr)\bigr) = \bigl\|\bigl(H_N(x,\omega) - E\bigr)^{-1}\bigr\|^{-1},$$

the lemma follows from Proposition 2.7. □

Next, we derive an important application of Lemma 2.5 and Proposition 2.12 to the Dirichlet determinants f_N. The constants C_0, C_1, C_2 depend on ω as explained above, see Definition 2.6.

Corollary 2.18. *Suppose $\omega \in \mathbb{T}_{c,a}$. Given $E_0 \in \mathbb{C}$ and $H > (\log N)^{C_2}$, $N \ge 2$, there exists*

$$\mathcal{B}_{N,E_0,\omega}(H) \subset \mathbb{C}, \qquad \mathcal{B}_{N,E_0,\omega}(H) \in \mathrm{Car}_1(\sqrt{H}, HN^2)$$

such that for any $z \in \mathbb{C} \setminus \mathcal{B}_{N,E_0,\omega}(H)$ with $|\mathrm{Im} z| < N^{-1}$, and large N the following holds: If

$$\log\bigl|f_N\bigl(e(z),\omega,E_1\bigr)\bigr| < NL(\omega,E_1) - H(\log N)^{C_2}, \quad |E_0 - E_1| < \exp(-(\log N)^{C_2}),$$

then $f_N\bigl(e(z),\omega,E\bigr) = 0$ for some $|E - E_1| \lesssim \exp(-\sqrt{H})$. Similarly, given $x_0 \in \mathbb{T}$ and $|y_0| < N^{-1}$, let $z_0 = e(x_0 + iy_0)$. Then for any $H \gg 1$, the following holds: if

$$\log\bigl|f_N\bigl(z_0,\omega,E\bigr)\bigr| < NL(\omega,E) - H(\log N)^{C_2},$$

then $f_N\bigl(z,\omega,E\bigr) = 0$ for some $|z - z_0| \lesssim \exp(-H)$.

Proof. Set $r_0 = \exp(-(\log N)^{C_0})$ with some (large) constant $C_0 = C_0(a)$ as above. Fix any z_0 with $|z_0| = 1$ and consider the analytic function

$$f(z,E) = f_N(z_0 + (z - z_0)N^{-1}, E_0 + (E - E_0)r_0, \omega)$$

on the polydisk $\mathcal{P} = \mathcal{D}(z_0,1) \times \mathcal{D}(E_0,1)$. Then, by Proposition 2.12,

$$\sup_{\mathcal{P}} \log|f(z,E)| \le NL(E_0,\omega) + C(\log N)^{C_0} = M$$

and by the large deviation theorem,

$$\log|f(z_1,E_0)| > NL(E_0,\omega) - (\log N)^{C_0} = m$$

for some $|z_0 - z_1| < 1/100$, say. By Lemma 2.5 there exists

$$\mathcal{B}_{z_0,E_0,\omega}(H) \subset \mathbb{C}, \qquad \mathcal{B}_{z_0,E_0,\omega}(H) \in \mathrm{Car}_1(\sqrt{H}, H(\log N)^{C_1})$$

so that for any $z \in \mathcal{D}(z_0,1/2) \setminus \mathcal{B}_{z_0,E_0,\omega}(H)$ the following holds: If

$$\log|f(z,E_1)| < NL(E_0,\omega) - H(\log N)^{C_1}$$

for some $|E_1 - E_0| < 1/2$, then there is E with $|E_1 - E| \lesssim \exp(-\sqrt{H})$ such that $f(z,E) = 0$. Now let z_0 run over a $N^{-\frac{3}{2}}$-net on $|z| = 1$ and define $\mathcal{B}_{N,E_0,\omega}(H)$ to be the union of the sets $z_0 + N^{-1}\mathcal{B}_{z_0,E_0,\omega}(H)$. The first half of the lemma now follows by taking C_2 sufficiently large and by absorbing some powers of $\log N$ into H if needed.

The second half of the lemma dealing with zeros in the z variable can be shown without appealing to Lemma 2.5. Indeed, we apply Cartan's estimate in $d = 1$ directly to $u(\cdot) = \log|f_N(\cdot,\omega,E)|$ on the disk $\mathcal{D}(z_0,N^{-1})$. By the preceding the Riesz mass of $u(\cdot)$ on this disk is at most $(\log N)^{C_0}$. Hence, we can find a radius $r \asymp \exp(-H)$ so that

$$\min_{|z-z_0|=r} \log\big|f_N\big(z,\omega,E\big)\big| > NL(\omega,E) - H(\log N)^{C_2}$$

Now if

$$\log\big|f_N\big(z_0,\omega,E\big)\big| < NL(\omega,E) - H(\log N)^{C_2},$$

then from the maximum principle $f_N\big(z_0,\omega,E\big) = 0$ for some $|z - z_0| < r$ as claimed. □

Corollary 2.18 should be thought of as a *converse* to the large deviation theorem in some sense; indeed, it shows that if $\log|f_N|$ is too small at some point, then nearby there must be a zero. In other words, and not surprisingly, zeros are responsible for the failure of the large deviation estimates.

The following result allows us to translate separations of an energy E_0 from the spectrum of $H_N(x,\omega)$ into quantitative lower bounds on $\log|f_N(x,\omega,E_0)|$. For it we need V to be real-valued on $\mathbb{T}$. As usual, $\omega \in \mathbb{T}_{c,a}$, and we remind the reader that C_1, C_2 etc. depend on ω, see Definition 2.6. Before proving it, we recall a basic fact of Hermitian matrices. It will be applied repeatedly in this paper.

Lemma 2.19. *Suppose A is a Hermitian $n \times n$-matrix. Further, let B be another $n \times n$-matrix with $\|A - B\| < \varepsilon$ in operator norm. Then*

$$\mathrm{dist}(\mathrm{spec}(A), \mathrm{spec}(B)) < \varepsilon$$

Proof. Suppose $z \in \mathbb{C}$ satisfies $\mathrm{dist}(z, \mathrm{spec}(A)) \geq \varepsilon$. Since A is Hermitian, we see (from the spectral theorem) that

$$\|(A - z)^{-1}\| \leq \varepsilon^{-1}$$

Then

$$R(z) := \sum_{n=0}^{\infty} (A - B)^n (A - z)^{-(n+1)}$$

converges as a Neuman series in operator norm. Moreover, $R(z)(B - z) = (B - z)R(z) = I$, the identity. Hence $z \in \mathbb{C} \setminus \mathrm{spec}(B)$ whence the lemma. □

We can now state another important type of converse of the large deviation theorem.

Corollary 2.20. *Let V be real-valued on $\mathbb{T}$. Assume that for sufficiently large N, $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$ one has*

$$\big(E_0 - \eta, E_0 + \eta\big) \cap \mathrm{spec}(H_N(x_0,\omega)) = \emptyset$$

with $\eta \leq \exp\big(-\big(\log N\big)^{C_1}\big)$. Then

$$\log\big|f_N\big(e(x_0),\omega,E\big)\big| > NL(\omega,E_0) - (\log N)^{C_1} \log\frac{1}{\eta}$$

for any $|E_0 - E| \le \frac{\eta}{2}$.

Proof. Suppose that

$$\log \big| f_N\big(e(x_0), \omega, E_1\big)\big| < NL(\omega, E_0) - (\log N)^{C_1} \log \eta^{-1}$$

for some $|E_1 - E_0| \le \frac{\eta}{2}$. Then there is $z_1 \in \mathbb{C}$ with $|z_1 - x_0| \ll \eta$ so that

$$f_N\big(e(z_1), \omega, E_1\big) = 0$$

Since $H_N(x, \omega)$ is Hermitian for $x \in \mathbb{T}$, it follows from Lemma 2.19 that the eigenvalues $E_j^{(N)}(\cdot, \omega)$ satisfy

$$|E_j^{(N)}(z, \omega) - E_j^{(N)}(x_0, \omega)| \le C|x_0 - z| \quad \forall\, z \in \mathcal{A}_{\rho_0/2}$$

In other words, there is some E_2 with $|E_2 - E_0| < \eta$ such that

$$f_N\big(e(x_0), \omega, E_2\big) = 0$$

However, this contradicts our assumption. □

We now address the important issue of a large deviation estimate with regard to the E variable.

Lemma 2.21. *Let $\omega_0 \in \mathbb{T}_{c,a}$ and $x_0 \in \mathbb{T}$. Then there exists $x_1 \in \mathbb{T}$ so that*

$$|x_1 - x_0| < \exp(-(\log N)^{C_0})$$
$$\mathrm{dist}\,\big(\,\mathrm{spec}(H_N(x_1, \omega_0)),\, \mathrm{spec}(H_N(x_0, \omega_0))\big) > \exp(-(\log N)^{C_1})$$

where $C_0 < C_1$.

Proof. Write $\mathrm{spec}(H_N(x_0, \omega_0)) = \{E_j(x_0, \omega_0)\}_{j=1}^N$. By Lemma 2.17,

$$\mathrm{mes}\,\big\{x \in \mathbb{T} \,:\, \min_{1 \le j \le N} \mathrm{dist}\big(\mathrm{spec}(H_N(x, \omega)), E_j(x_0, \omega_0)\big) < \exp(-(\log N)^{C_1})\big\} \lesssim \exp\big(-(\log N)^{C_0}\big)$$

where $C_0 < C_1$, and we are done. □

Lemma 2.22. *Let $\omega_0 \in \mathbb{T}_{c,a}$ and fix $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$. There exists $|E_1 - E_0| < \exp(-(\log N)^{C_0})$ with*

$$\log |f_N(e(x_0), \omega_0, E_1)| > NL(E_0, \omega_0) - (\log N)^{C_2} \tag{2.19}$$

where $C_2 > C_0$.

Proof. If

$$\mathrm{dist}(E_0, \mathrm{spec}(H_N(x_0, \omega_0))) > \exp(-(\log N)^{C_1})$$

then

$$\log |f_N(e(x_0), \omega_0, E_0)| > NL(E_0, \omega_0) - (\log N)^{2C_1}$$

by Corollary 2.20. Hence, in this case we can choose $E_1 = E_0$. Now assume that

$$\mathrm{dist}(E_0, \mathrm{spec}(H_N(x_0, \omega_0))) \le \exp(-(\log N)^{C_1}),$$

By the previous lemma we choose $|x_1 - x_0| < \exp(-(\log N)^{C_0})$ such that

$$\mathrm{dist}\,\big(\,\mathrm{spec}(H_N(x_1, \omega_0)),\, \mathrm{spec}(H_N(x_0, \omega_0))\big) > \exp(-(\log N)^{C_1}) \tag{2.20}$$

By self-adjointness, there exists $E_1 \in \mathrm{spec}(H_N(x_1, \omega_0))$ with

$$|E_1 - E_0| < C \exp(-(\log N)^{C_0})$$

which, in view of (2.20) also satisfies

$$\mathrm{dist}\,\big(\,\mathrm{spec}(H_N(x_0, \omega_0)),\, E_1\big) > \exp(-(\log N)^{C_1})$$

By Corollary 2.20 we conclude that

$$\log |f_N(e(x_0), \omega_0, E_1)| > NL(E_0, \omega_0) - (\log N)^{2C_1}$$

The lemma follows with $C_2 = 2C_1$. □

We can now state the large deviation estimate with respect to the E-variable.

Proposition 2.23. *Let $\omega_0 \in \mathbb{T}_{c,a}$ and assume that $L(\omega_0, E) > \gamma > 0$ for all $E \in [E', E'']$. Then for large N, and all $x_0 \in \mathbb{T}$,*

$$\operatorname{mes}\{E \in [E', E''] \,:\, |\log|f_N(e(x_0), \omega_0, E)| - NL(\omega_0, E)| > H\} \le C\exp(-H/(\log N)^{C_1}) \tag{2.21}$$

for all $H > (\log N)^{2C_1}$.

Proof. Let C_0 be as in the previous lemma. Covering $[E', E'']$ by intervals of length $100\exp(-(\log N)^{C_0})$ we see that it suffices to prove (2.21) locally on such an interval. Thus, consider a disk $\mathcal{D}(E_0, r_0)$ where $r_0 = 100\exp(-(\log N)^{C_0})$. By Lemma 2.22 there exists $E_1 \in \mathcal{D}(E_0, r_0/100)$ with

$$\log|f_N(e(x_0), \omega_0, E_1)| > NL(E_0, \omega_0) - (\log N)^{C_2}$$

On the other hand, there is the uniform upper bound

$$\sup_{E \in \mathcal{D}(E_0, r_0/100)} \log|f_N(e(x_0), \omega_0, E_1)| \le NL(E_0, \omega_0) + (\log N)^{C_2}$$

see Corollary 2.13. Now the proposition follows from Cartan's estimate. □

Remark 2.24. *Even though (2.21) was stated for real E, one can pass to a version of this estimate in the complex plane via Cartan's theorem: for all $H > (\log N)^{3C_1}$ there exist disks $\{\mathcal{D}(\zeta_j, r_j)\}_{j=1}^J$ with $\sum_j r_j < \exp(-H(\log N)^{-2C_1})$, $J \le (\log N)^{C_2}$ and*

$$\{E \in [E', E''] + \mathcal{D}(0, N^{-1}) \,:\, |\log|f_N(e(x_0), \omega_0, E)| - NL(\omega_0, E)| > H\} \subset \bigcup_j \mathcal{D}(\zeta_j, r_j)$$

for large N. This follows from Proposition 2.23 by choosing $H = (\log N)^{2C_1}$ (where C_1 is large depending on (E', E'')) which insures that there is at least one energy in (E', E'') satisfying (2.21). Now apply Cartan's theorem as in part (a) of this section.

We close this subsection with an important consequence of the previous estimates; it allows us to bounds the number of zeros of the determinants with respect to both the z and E variables.

Proposition 2.25. *Let V be analytic on $\mathcal{A}_{\rho_0}$ and real-valued on $\mathbb{T}$. Let $\omega \in \mathbb{T}_{c,a}$. Then for any $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$ one has*

$$\#\big\{E \in \mathbb{R} \,:\, f_N\big(e(x_0), \omega, E\big) = 0,\ |E - E_0| < \exp\big(-(\log N)^{C_1}\big)\big\} \le (\log N)^{C_1} \tag{2.22}$$

$$\#\big\{z \in \mathbb{C} \,:\, f_N(z, \omega, E_0) = 0,\ |z - e(x_0)| < N^{-1}\big\} \le (\log N)^{C_1} \tag{2.23}$$

for all sufficiently large $N \ge N(V, \gamma, \rho_0, \omega, E_0)$.

Proof. By the uniform upper bound

$$\sup\big\{\log\big|f_N(e(x), \omega, E)\big| \,:\, x \in \mathbb{T},\ E \in \mathbb{C},\ |E - E_1| < \exp\big(-(\log N)^{C_1}\big)\big\} \le NL_N(\omega, E_1) + (\log N)^{C_1}$$

for any E_1. Due to the large deviation theorem with respect to the E variable, see Proposition 2.23, there exist x_1, E_1 such that $|x_0 - x_1| < \exp\big(-(\log N)^{2C_1}\big)$, $|E_0 - E_1| < \exp\big(-(\log N)^{2C_1}\big)$ so that

$$\log\big|f_N(e(x_1), \omega, E_1)\big| > NL_N(\omega, E_1) - (\log N)^{C_1}.$$

By Jensen's formula (4.1),

$$\#\big\{E \,:\, f_N(e(x_1), \omega, E) = 0, |E - E_1| < \exp\big(-(\log N)^{C_1}\big)\big\} \le 2(\log N)^{C_1}.$$

Since $\big\|H_N(x_0, \omega) - H_N(x_1, \omega)\big\| \lesssim \exp\big(-(\log N)^{2C_1}\big)$ and since $H_N(x_0, \omega)$ is Hermitian one has

$$\begin{aligned}&\#\big\{E : f_N(e(x_0), \omega, E) = 0, |E - E_0| < \exp\big(-(\log N)^{2C_1}\big)\big\}\\ &\le \#\big\{E : f_N(e(x_1), \omega, E) = 0, |E - E_1| < \exp\big(-(\log N)^{C_1}\big)\big\} \le (\log N)^{C_1}\ .\end{aligned}$$

That proves (2.22). The proof of (2.23) is similar. Indeed, due to the uniform upper bound

$$\sup\big\{\log\big|f_N(e(x + iy), \omega, E_0)\big| \,:\, x \in \mathbb{T},\ |y| < 2N^{-1}\big\} \le NL_N(\omega, E_0) + (\log N)^{C_1}.$$

By the large deviation theorem, there is x_1 with $|x_0 - x_1| < \exp(-(\log N)^{C_1/2})$ such that

$$\log \big| f_N(e(x_1), \omega, E_0) \big| > N L_N(\omega, E_0) - (\log N)^{C_1}$$

Hence, by Jensen's formula (4.1),

$$\#\big\{ z \,:\, f_N(z, \omega, E) = 0, |z - e(x_1)| < 2N^{-1} \big\} \leq 2(\log N)^{C_1},$$

and (2.23) follows. □

(g) The Weierstrass preparation theorem for Dirichlet determinants

Recall the Weierstrass preparation theorem for an analytic function $f(z, w_1, \ldots, w_d)$ defined in a polydisk

$$\mathcal{P} = \mathcal{D}(z_0, R_0) \times \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, R_0), \quad z_0,\ w_{j,0} \in \mathbb{C} \qquad \frac{1}{2} \geq R_0 > 0\ . \tag{2.24}$$

Proposition 2.26. *Assume that $f(\cdot, w_1, \ldots, w_d)$ has no zeros on some circle $\big\{ z : |z - z_0| = \rho_0 \big\}$, $0 < \rho_0 < R_0/2$, for any $\underline{w} = (w_1, \ldots, w_d) \in \mathcal{P}_1 = \prod_{j=1}^{d} \mathcal{D}(w_{j,0}, r_1)$ where $0 < r_1 < R_0$. Then there exist a polynomial $P(z, \underline{w}) = z^k + a_{k-1}(\underline{w}) z^{k-1} + \cdots + a_0(\underline{w})$ with $a_j(\underline{w})$ analytic in $\mathcal{P}_1$ and an analytic function $g(z, \underline{w}), (z, \underline{w}) \in \mathcal{D}(z_0, \rho_0) \times \mathcal{P}_1$ so that the following properties hold:*

(a) *$f(z, \underline{w}) = P(z, \underline{w}) g(z, \underline{w})$ for any $(z, \underline{w}) \in \mathcal{D}(z_0, \rho_0) \times \mathcal{P}_1$.*
(b) *$g(z, \underline{w}) \neq 0$ for any $(z, \underline{w}) \in \mathcal{D}(z_0, \rho_0) \times \mathcal{P}_1$*
(c) *For any $\underline{w} \in \mathcal{P}_1$, $P(\cdot, \underline{w})$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, \rho_0)$.*

Proof. By the classical Weierstrass argument,

$$b_p(\underline{w}) := \sum_{j=1}^{k} \zeta_j^p(\underline{w}) = \frac{1}{2\pi i} \oint_{|z - z_0| = \rho_0} z^p\ \frac{\partial_z f(z, \underline{w})}{f(z, \underline{w})}\, dz$$

are analytic in $\underline{w} \in \mathcal{P}_1$. Here $\zeta_j(\underline{w})$ are the zeros of $f(\cdot, \underline{w})$ in $\mathcal{D}(z_0, \rho_0)$ counted with multiplicity. Since the coefficients $a_j(\underline{w})$ are linear combinations of the b_p, they are analytic in $\underline{w}$. Analyticity of g follows by standard arguments. □

Since there is an estimate for the local number of the zeros of the Dirichlet determinant and also the local number of the Dirichlet eigenvalues, one can apply Proposition 2.26 to $f_N(z, \omega, E)$. We need to do this in both the z and the E variables. See Section 6 of [GolSch2] for more details. In what follows recall the convention adopted in Definition 2.6.

Proposition 2.27. *Given $z_0 \in \mathcal{A}_{\rho_0/2}$, $E_0 \in \mathbb{C}$, and $\omega_0 \in \mathbb{T}_{c,a}$, there exists $N_0 = N_0(V, \rho_0, a, c, \gamma)$ so that the following holds: for any $N \geq N_0$ there exists a polynomial*

$$P_N(z, \omega, E) = z^k + a_{k-1}(\omega, E) z^{k-1} + \cdots + a_0(E, \omega)$$

with $a_j(\omega, E)$ analytic in $\mathcal{D}(E_0, r_1) \times \mathcal{D}(\omega_0, r_1)$, $r_1 \asymp \exp\big(-(\log N)^{C_1}\big)$ and an analytic function

$$g_N(z, \omega, E), \quad (z, \omega, E) \in \mathcal{P} := \mathcal{D}(z_0, r_0) \times \mathcal{D}(E_0, r_1) \times \mathcal{D}(\omega_0, r_1)$$

with $N^{-1} \leq r_0 \leq 2N^{-1}$ such that:

(a) *$f_N(z, \omega, E) = P_N(z, \omega, E) g_N(z, \omega, E)$*
(b) *$g_N(z, \omega, E) \neq 0$ for any $(z, \omega, E) \in \mathcal{P}$*
(c) *For any $(\omega, E) \in \mathcal{D}(\omega_0, r_1) \times \mathcal{D}(E_0, r_1)$, the polynomial $P_N(\cdot, \omega, E)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, r_0)$*
(d) *$k = \deg P_N(\cdot, \omega, E) \leq (\log N)^{C_0}$.*

Proof. With $r_0 := 2N^{-1}$ and $r_1 := \exp(-(\log N)^{C_1})$, we set

$$f(\zeta, w_1, w_2) := f_N(z_0 + N^{-1}\zeta, \omega_0 + r_1 w_1, E_0 + r_1 w_2) \quad \forall\, (\zeta, w_1, w_2) \in \mathcal{D}(0,1)^3$$

Then by the uniform upper bound $|f| \le \exp(NL(\omega_0, E_0) + (\log N)^{C_0}) =: M$ on $\mathcal{D}(0,1)^3$ and, by the large deviation theorem,

$$|f(\zeta, 0, 0)| > \exp\big(NL(\omega_0, E_0) - (\log N)^{C_0}\big)$$

for all $|\zeta| = r$ and some $\frac{1}{2} < r < 1$. Moreover, by Cauchy's estimate

$$|f(\zeta, 0, 0) - f(\zeta, w_1, w_2)| \le 2M(|w_1| + |w_2|) \le \frac{M}{2} \exp(-2(\log N)^{C_0})$$

for all $|w_1| + |w_2| < \frac{1}{4}\exp(-2(\log N)^{C_0})$. In particular,

$$f(\zeta, w_1, w_2) \neq 0 \quad \forall\, |\zeta| = r,\ |w_1| + |w_2| < \frac{1}{4}\exp(-2(\log N)^{C_0})$$

The proposition follows by applying Proposition 2.26 and a rescaling. □

Later we shall need to localize Proposition 2.27 to smaller regions in E and z.

Corollary 2.28. *Using the notations of the previous proposition, let $0 < r_2 < e^{-(\log N)^{C_1}}$ be given. With the same hypotheses, the conclusions of Proposition 2.27 hold on the smaller poly-disk*

$$\mathcal{D}(z_0, r_2') \times \mathcal{D}(E_0, r_2'') \times \mathcal{D}(\omega_0, r_2'')$$

where

$$r_2' \asymp r_2, \qquad r_2'' \asymp r_2^{3(\log N)^{C_1}}$$

Proof. Apply the proposition and let $z_j(E, \omega)$ be the zeros of $P_N(\cdot, \omega, E)$. Then

$$P_N(z, \omega, E) = \prod_{j=1}^{k} (z - z_j(E, \omega))$$

Select r_2' so that

$$\inf_{|z - z_0| = r_2'} |P_N(z, \omega, E)| \ge \big(r_2/(\log N)^{C_1}\big)^{(\log N)^{C_1}}$$

Since $|a_j(\omega, E)| \le 1$, it follows that

$$\inf_{\substack{|E-E_0| \le r_2'' \\ |\omega - \omega_0| \le r_2''}} \ \inf_{|z - z_0| = r_2'} |P_N(z, \omega, E)| \ge \frac{1}{2}\big(r_2/(\log N)^{C_1}\big)^{(\log N)^{C_1}} \ge r_2^{2(\log N)^{C_1}}$$

where r_2'' is as above. We can now apply Proposition 2.26 as before. □

The preparation theorem relative to E is easier since we need it only in the neighborhood of the unit circle, i.e., in the neighborhood of points $e(x_0)$ with $x_0 \in \mathbb{T}$. In this case, one can use the fact that $H_N(e(x_0), \omega)$ is Hermitian.

Proposition 2.29. *Given $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{C}$, and $\omega_0 \in \mathbb{T}_{c,a}$, there exist a polynomial*

$$P_N(z, \omega, E) = E^k + a_{k-1}(z, \omega) E^{k-1} + \cdots + a_0(z, \omega)$$

with $a_j(z, \omega)$ analytic in $\mathcal{D}(z_0, r_1) \times \mathcal{D}(\omega_0, r_1)$, $z_0 = e(x_0)$, $r_1 \asymp \exp\big(-(\log N)^{2C_1}\big)$ and an analytic function $g_N(z, \omega, E)$, $(z, \omega, E) \in \mathcal{P} = \mathcal{D}(z_0, r_1) \times \mathcal{D}(\omega_0, r_1) \times \mathcal{D}(E_0, r_1)$ such that

(a) $f_N(z, \omega, E) = P_N(z, \omega, E) g_N(z, \omega, E)$
(b) $g_N(z, \omega, E) \neq 0$ *for any* $(z, \omega, E) \in \mathcal{P}$
(c) *For any $(z, \omega) \in \mathcal{D}(z_0, r_1) \times \mathcal{D}(\omega_0, r_1)$, the polynomial $P_N(z, \omega, \cdot)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(E_0, r_0)$, $r_0 \asymp \exp\big(-(\log N)^{C_1}\big)$*
(d) $k = \deg P_N(z, \omega, \cdot) \le (\log N)^{C_2}$

Proof. Recall that due to Proposition 2.25 one has

$$\#\big\{E \in \mathbb{C} : f_N(z_0, \omega_0, E) = 0, \quad |E - E_0| < \exp\big(-(\log N)^{C_1}\big)\big\} \le (\log N)^{C_1}$$

Find $r_0 \asymp \exp\big(-(\log N)^{C_1}\big)$ such that $f_N(z_0, \omega_0, \cdot)$ has no zeros in the annulus

$$\big\{r_0(1 - 2N^{-2}) < |E - E_0| < r_0(1 + 2N^{-2})\big\}.$$

Since $H_N(z_0, \omega_0)$ is self-adjoint, $f_N(z, \omega, \cdot)$ has no zeros in the annulus

$$\big\{r_0(1 - N^{-2}) < |E - E_0| < r_0(1 + N^{-2})\big\},$$

provided $|z - z_0| \ll r_1 := r_0 N^{-4}$, $|\omega - \omega_0| \ll r_1$, see Lemma 2.19. The proposition now follows from Proposition 2.26. □

3. The trace of M_N and Hill's discriminant of the periodic problem

This section establishes large deviation estimates for the trace of M_N as well as other useful relations involving the trace. The importance of this section, which does not appear in [GolSch2], lies with periodic boundary conditions: recall that the determinant of the Hamiltonian $H_{[1,N]}$ with periodic boundary conditions equals the trace of the Monodromy matrix M_N up to a constant (the latter trace is referred to as "Hill's discriminant"). In our proof of gap formation periodic boundary conditions play an important technical role, whence the relevance of this section. Let us recall some properties of matrices in $SL(2, \mathbb{R})$. It follows from the polar decomposition that for any $M \in SL(2, \mathbb{R})$ there are unit vectors $\underline{u}_M^+$, $\underline{u}_M^-$, $\underline{v}_M^+$, $\underline{v}_M^-$ so that $M\underline{u}_M^+ = \|M\|\underline{v}_M^+$, $M\underline{u}_M^- = \|M\|^{-1}\underline{v}_M^-$. Moreover, $\underline{u}_M^+ \perp \underline{u}_M^-$ and $\underline{v}_M^+ \perp \underline{v}_M^-$.

Lemma 3.1. *For any $M \in SL(2, \mathbb{R})$,*

$$\|M^2\| - 4 \le \|M\|\,|\operatorname{tr} M| \le \|M^2\| + 2. \tag{3.1}$$

Proof. Due to the properties of the vectors $\underline{u}^+ = \underline{u}_M^+$, $\underline{u}^- = \underline{u}_M^-$ one has

$$\operatorname{tr} M = \|M\|\underline{v}^+ \cdot \underline{u}^+ + \|M\|^{-1}\underline{v}^- \cdot \underline{u}^- \tag{3.2}$$

On the other hand,

$$M^2\underline{u}^+ = \|M\|^2(\underline{u}^+ \cdot \underline{v}^+)\underline{v}^+ + (\underline{v}^+ \cdot \underline{u}^-)\underline{v}^-. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$|\operatorname{tr} M|\,\|M\| \le \|M\|^2\,|\underline{u}^+ \cdot \underline{v}^+| + 1 \le \|M^2\| + 2,$$

as well as

$$|\operatorname{tr} M|\,\|M\| \ge \|M\|^2\,|\underline{u}^+ \cdot \underline{v}^+| - 1 \ge \|M^2\underline{u}^+\| - 2.$$

Finally, using that $\|M\| \ge 1$ one checks that $\|M^2\underline{u}^-\| \le 2$, and thus $\|M^2\underline{u}^+\| \ge \|M^2\| - 2$. Inserting this bound into the last line finishes the proof. □

The following lemma establishes the large deviation estimate for a product of mondromy matrices. The technical (albeit, important) twist here is that we shift the phase in the second factor by a small but *fixed* amount. This will be essential for applications to the trace. Indeed, in view of the previous lemma, in order to prove a large deviation theorem for tr M_N, say, it will be necessary to do the same for M_N^2. The latter should behave like M_{2N}, but more precisely it is equal to $M_N(x + N\omega + \kappa)M_N(x)$ where $\kappa \equiv -N\omega \pmod 1$.

Lemma 3.2. *Assume that for some $\omega \in \mathbb{T}_{c,a}$ and $E \in \mathbb{R}$, one has $L(\omega, E) \ge \gamma > 0$. Then there exists $\kappa_0 = \kappa_0(V, \omega, \gamma, E) > 0$ such that for any $|\kappa| \le \kappa_0$*

$$\begin{aligned} &\operatorname{mes}\big\{x \in \mathbb{T} \,:\, \big|\log\big\|M_N\big(e(x + N\omega + \kappa), \omega, E\big)M_N\big(e(x), \omega, E\big)\big\| - 2NL(E, \omega)\big| > H\big\} \\ &\le C\exp\big(-H/\big(\log N\big)^{C_2}\big) \end{aligned} \tag{3.4}$$

for all $H > 0$ and $N \ge 2$.

Proof. This will be done by induction in N; more precisely, we will introduce an increasing integer sequence $\{N_j\}_{j\geq 0}$ so that if (3.4) holds for all $N_j \leq N < N_{j+1}$, then it also holds in the range $N_{j+1} \leq N < N_{j+2}$. Clearly, by choosing $N_0 := N_0(\gamma, V, \omega, E)$ large and $\kappa_0 := \exp(-CN_1)$ we see that the case $j = 0$ can be made to hold for any N_1. Next, let $N_{j+1} \leq N < N_{j+2}$ and set $n := [(\log N)^{C_1}]$ where $C_1 = 2C_0$ with C_0 as in (2.4). Also, we define $N_{j+1} := \exp\big(N_j^{\frac{1}{2C_1}}\big)$. By the large deviation theorem from Section 2 as well as our inductive assumption (applied with $H = n^{2/3}$, say), there is an avalanche principle expansion of the form

$$\begin{aligned}
&\log \big\|M_N\big(e(x+N\omega+\kappa),\omega,E\big)M_N\big(e(x),\omega,E\big)\big\| - \log\big\|M_N\big(e(x+N\omega+\kappa),\omega,E\big)\big\| \\
&\quad - \log\big\|M_N\big(e(x),\omega,E\big)\big\| \\
&= \log\|M_n\big(e(x+N\omega+\kappa),\omega,E\big)M_n\big(e(x+(N-n)\omega),\omega,E\big)\| - \log\|M_n\big(e(x+N\omega+\kappa),\omega,E\big)\| \\
&\quad - \log\|M_n\big(e(x+(N-n)\omega),\omega,E\big)\| + O(e^{-\sqrt{n}})
\end{aligned}$$

for all $x \in \mathbb{T}\setminus\mathcal{B}$ where $\operatorname{mes}(\mathcal{B}) < e^{-\sqrt{n}}$. In particular, this implies that

$$\log\big\|M_N\big(e(x_1+N\omega+\kappa),\omega,E\big)M_N\big(e(x_1),\omega,E\big)\big\| > 2NL(\omega,E) - (\log N)^{C_1}$$

for all $x_1 \in \mathbb{T}\setminus\mathcal{B}$. Next, note from the uniform upper bounds in Part (e) of Section 2 that

$$\sup_{x\in\mathbb{T},\,|y|<N^{-1}} \log\big\|M_N\big(e(x+iy+N\omega+\kappa),\omega,E\big)M_N\big(e(x+iy),\omega,E\big)\big\| < 2NL(\omega,E) + (\log N)^{C_1}$$

By averaging, we conclude that

$$\Big|\int_0^1 \log\big\|M_N\big(e(x+iy+N\omega+\kappa),\omega,E\big)M_N\big(e(x+iy),\omega,E\big)\big\|\,dx - 2NL(\omega,E)\Big| < (\log N)^{2C_1}$$

Furthermore, by Cartan's theorem, see Lemma 2.4, this yields (3.4) for N with $C_2 = 4C_1$, say. □

We can now state and prove the main result of this section. Note that even the average of $\log|\operatorname{tr} M_N|$, which appears as the first statement below, is far from being clear and requires much of the machinery developed so far in this paper.

Proposition 3.3. *Assume that for some $\omega \in \mathbb{T}_{c,a}$, $E \in \mathbb{R}$ one has $L(\omega,E) \geq \gamma > 0$ and let $\kappa_0 > 0$ be as in the previous lemma. Then there exists $N_0 = N_0(V,c,a,E,\gamma)$ such that for any $N \geq N_0$ satisfying $\|N\omega\| < \kappa_0$ the following properties hold:*

- $\int_{\mathbb{T}} \log|\operatorname{tr} M_N(e(x),\omega,E)|\,dx = NL(\omega,E) + O(1)$
- Large deviation estimate for the traces:

$$\operatorname{mes}\big\{x\in[0,1] : |\log|\operatorname{tr} M_N(e(x+iy),\omega,E)| - NL(\omega,E)| > H\big\} \leq C\exp(-H(\log N)^{-C_0})$$

  for all $|y| < N^{-1}$ and all $H > 0$.

Proof. For simplicity, we set $y = 0$. We begin with the simple observation that

$$\log\|M_N(e(x),\omega,E)^2\| = \log\|M_N(e(x+N\omega+\kappa),\omega,E)M_N(e(x),\omega,E)\| \tag{3.5}$$

where $\kappa \equiv -N\omega \pmod 1$. By assumption, we can choose $|\kappa| < \kappa_0$ from Lemma 3.2. We apply the avalanche principle to $M_N(e(x+N\omega+\kappa),\omega,E)M_N(e(x),\omega,E)$. To this end, define for $1 \leq j \leq m$

$$A_j = M_{n_j}(e(x+t_j\omega),\omega,E),\ (\log N)^{C_0} < n_j < \sqrt{N},\ t_j = \sum_{i<j} n_i,\ \sum_{i=1}^m n_i = N$$

as well as $A_{j+m}(x) := A_j(x + N\omega + \kappa) = A_j(x)$. We also require that $n_1 = n_m$. Then by the large deviation estimate (2.4) and Lemma 3.2, we conclude that

$$
\begin{aligned}
\log \|M_N(e(x),\omega,E)^2\| &= \sum_{j=1}^{2m-1} \log \|A_{j+1}(x)A_j(x)\| - \sum_{j=1}^{2m-1} \log \|A_j(x)\| + 0(e^{-\sqrt{\underline{n}}}) \\
&= 2\Big[\sum_{j=1}^{m-1} \log \|A_{j+1}(x)A_j(x)\| - \sum_{j=1}^{m-1} \log \|A_j(x)\|\Big] + \\
&\quad + \log\|A_1(x)A_m(x)\| - \log\|A_1(x)\| - \log\|A_m(x)\| + 0(e^{-\sqrt{\underline{n}}})
\end{aligned}
\tag{3.6}
$$

for any $x \in \mathbb{T} \setminus \mathcal{B}$, with mes $\mathcal{B} < \exp(-\underline{n}^{1/2})$ where $\underline{n} = \min_i n_i$. Interpreting the expression in brackets as expansion of $\log\|M_N(e(x),\omega,E)\|$, and in view of Lemma 3.1, we see that

$$
\begin{aligned}
\log|\,\mathrm{tr}\; M_N(e(x),\omega,E)| - \log\|M_N(e(x),\omega,E)\| &= \log\|A_1(x)A_m(x)\| - \log\|A_1(x)\| \\
&\quad - \log\|A_m(x)\| + 0(e^{-\sqrt{\underline{n}}})
\end{aligned}
\tag{3.7}
$$

for any $x \in \mathbb{T} \setminus \mathcal{B}$ with the same $\mathcal{B}$. The proposition now follows from Lemma 3.2 and the standard large deviation theorem for the matrices M_N. $\square$

For future reference, we remark that the *avalanche principle expansion* of $\log|\,\mathrm{tr}\; M_N|$ given by (3.6), and the comparison statement (3.7) are of independent interest. Note that

$$
\log|\,\mathrm{tr}\; M_N(e(x),\omega,E)| \le \log\|M_N(e(x),\omega,E)\| + \log 2
$$

In particular, due to Proposition 2.12, one has the following uniform upper bound for the trace:

Corollary 3.4. *Assume $L(\omega,E) \ge \gamma > 0$, $\omega \in \mathbb{T}_{c,a}$. Then*

$$
\sup_{x\in\mathbb{T}} \log|\,\mathrm{tr}\; M_N(x,\omega,E)| \le NL_N(\omega,E) + C(\log N)^{C_0}\ ,
$$

for all $N \ge 2$, provided $\|N\omega\| \le \kappa_0(V,c,a,\gamma)$.

Now just as in Section 2 one has the following

Corollary 3.5. *Assume $L(\omega,E_1) \ge \gamma > 0$, $\omega_1 \in \mathbb{T}_{c,a}$. Then for any $\omega \in \mathbb{T}$, y_1, $y \in \mathbb{R}$, $|y_1|$, $|y| \le 1/N$, and any x_1,$x \in \mathbb{T}$ one has*

$$
\begin{aligned}
\Big|\log\frac{\big|\mathrm{tr}\; M_N\big(e(x+iy),\omega,E\big)\big|}{\big|\mathrm{tr}\; M_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\Big| &\le \big(|E-E_1| + |\omega-\omega_1| + |x-x_1| + |y-y_1|\big) \\
&\quad \frac{\exp\big(NL(y_1,\omega_1,E_1) + C(\log N)^{C_0}\big)}{\big|\mathrm{tr}\; M_N\big(e(x_1+iy_1),\omega_1,E_1\big)\big|}\ ,
\end{aligned}
\tag{3.8}
$$

provided $\|N\omega\| \le \kappa_0(V,c,a,\gamma)$ and the right-hand side of (3.8) is less than $1/2$.

With these results on the trace at our disposal, we now turn to their implications for the periodic problem. To fix notation, let

$$
\big[H^{(\pm P)}_{[1,N]}(x,\omega)\psi\big](n) = -\psi(n-1) - \psi(n+1) + V(e(x+n\omega))\psi(n) \tag{3.9}
$$

be the Schrödinger operator on $[1,N]$ with periodic (respectively, antiperiodic) boundary conditions.

$$
\psi(0) = \pm\psi(N),\ \psi(1) = \pm\psi(N+1) \tag{3.10}
$$

Let

$$
E_1^{(N,\pm P)}(x,\omega) \le E_2^{(N,\pm P)}(x,\omega) \le \cdots \le E_N^{(N,\pm P)}(x,\omega) \tag{3.11}
$$

be the eigenvalues of $H^{(\pm P)}_{[1,N]}(x,\omega)$. Recall that the characteristic determinant

$$
g_N^{(\pm)}(e(x),\omega,E) := \det\big(H^{(\pm P)}_{[1,N]}(x,\omega) - E\big)
$$

which takes the form

$$g_N^{(\pm)}(e(x),\omega,E) = \det \begin{bmatrix} V(e(x+\omega)-E & -1 & 0 & \dots & 0 & \mp 1 \\ -1 & V(e(x+2\omega)-E & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & & -1 \\ \mp 1 & 0 & \dots & \dots & -1 & V(e(x+N\omega)-E \end{bmatrix}$$

satisfies

$$g_N^{(\pm)}(e(x),\omega,E) = h_N(e(x),\omega,E) \mp 2 \tag{3.12}$$

where

$$h_N(e(x),\omega,E) := \mathrm{tr}\ M_N(e(x),\omega,E) = f_{[1,N]}(e(x),\omega,E) - f_{[2,N-1]}(e(x),\omega,E) \tag{3.13}$$

is Hill's discriminant. Cramer's rule then yields

$$(H_{[1,N]}^{(\pm P)}(x,\omega)-E)^{-1}(m,n) = \frac{f_{[1,m-1]}(e(x),\omega,E)f_{[n+1,N]}(e(x),\omega,E)}{g_N^{(\pm)}(e(x),\omega,E)},\ 1\le m\le n\le N \tag{3.14}$$

The large deviation estimate for $\log|\,\mathrm{tr}\ M_N(e(x),\omega,E)|$ from above implies the following lemma concerning the spectrum of the periodic problem. In particular, we obtain an analogue of the Wegner bound.

Lemma 3.6. *Let $V(e(x))$ be real analytic, and let $\omega \in \mathbb{T}_{c,a}$. Assume that $L(\omega,E)\ge\gamma>0$ for some $E\in\mathbb{R}$. Then there exists $N_0 = N_0(V,a,c,\gamma,E)$ such that for any $N\ge N_0$ with $\|N\omega\|\le\kappa_0(V,c,a,\gamma)$ the following properties hold:*

(a) *For any $H>(\log N)^{2C_0}$ one has*

$$\mathrm{mes}\,\{x\in\mathbb{T} : \mathrm{dist}(\mathrm{spec}(H_{[1,N]}^{(\pm P)}(x,\omega)),E) < \exp(-H)\} \le \exp(-H/(\log N)^{C_0})$$

(b) *If for some $x\in\mathbb{T}$*

$$(E-\eta,E+\eta)\cap \mathrm{spec}\left(H_N^{(\pm P)}(x,\omega)\right) = \emptyset$$

with $\eta\le\exp(-(\log N)^{2C_0})$ then

$$\log|g_N^{(\pm P)}(e(x'),\omega,E')| > NL(E,\omega) - (\log N)^{C_1}\log\frac{1}{\eta}$$

for any $|x'-x|+|E'-E|<\eta/C$.

Proof. The proof of (a) is analogous to the proof of Lemma 2.17; the only difference is that we apply the large deviation theorem for the traces instead of for the determinants. Part (b) is analogous to Corollary 2.20, at least when $x'=x$, and we skip the proof. Finally, to move x to x' one uses Corollary 3.5. □

To close this section, we prove the following large deviation theorem for the traces with respect to the E variable, cf. Proposition 2.23. It will play an important role later when we start counting eigenvalues by means of the Jensen average machinery which is subject of the following section.

Proposition 3.7. *Let $\omega_0\in\mathbb{T}_{c,a}$ and assume that $L(\omega_0,E)>\gamma>0$ for all $E\in[E',E'']$. Then for large N, with $\|N\omega\|\le\kappa_0(V,c,a,\gamma)$ and all $x_0\in\mathbb{T}$,*

$$\mathrm{mes}\,\{E\in[E',E'']\ :\ |\log|g_N^{(\pm P)}(e(x_0),\omega_0,E)| - NL(\omega_0,E)| > H\} \le C\exp(-H/(\log N)^{C_1}) \tag{3.15}$$

for all $H>(\log N)^{2C_1}$. The same statement applies to $\mathrm{tr}\ M_N(e(x_0),\omega_0,\cdot)$.

Proof. This proof is completely analogous to the proof of Proposition 2.23. Indeed, the previous Lemma 3.6 replaces the Wegner type lemma used there as well as Corollary 2.20. □

4. Zeros, eigenvalues, and the Jensen formula

This section introduces a key element in our approach to the problem of determining the location of the spectrum and of the spectral gaps. More specifically, we identify the spectral values and the spectral gaps according to whether $f_N(\cdot,\omega,E)$ has a sequence of real or complex zeros in the annulus $\mathcal{A}_{\rho_N} = \big\{z \in \mathbb{C} : 1-\rho_N < |z| < 1+\rho_N\big\}$ with appropriate ρ_N. We would also like to single out Lemma 4.6 below. It guarantees that the gaps in finite volume stabilize after a finite (and uniformly bounded) number of iterations in our "induction on scales" procedure. An important feature of the machinery developed here lies with the fact that it applies equally well to the z-variable as to the E-variable of $f_N(z,\omega,E)$. Another important feature is the "linearity" of our bounds which means that the zero count is additive.

Now for the details, which for the most part already appear in [GolSch2] (with the exception of the crucial Lemma 4.6). The Jensen formula states that for any function f analytic on a neighborhood of $\mathcal{D}(z_0,R)$, see [Lev],

$$\int_0^1 \log|f(z_0+Re(\theta))|\,d\theta - \log|f(z_0)| = \sum_{\zeta:f(\zeta)=0} \log\frac{R}{|\zeta-z_0|} \tag{4.1}$$

provided $f(z_0)\neq 0$. In the previous section, we showed how to combine this fact with the large deviation theorem and the uniform upper bounds to bound the number of zeros of f_N which fall into small disks, in both the z and E variables. In what follows, we will refine this approach further. For this purpose, it will be convenient to average over z_0 in (4.1). Henceforth, we shall use the notation

$$\nu_f(z_0,r) = \#\{z\in\mathcal{D}(z_0,r) : f(z)=0\} \tag{4.2}$$

$$\mathcal{J}(u,z_0,r_1,r_2) = \fint_{\mathcal{D}(z_0,r_1)} \mathcal{J}(u,z,r)\,dx\,dy \tag{4.3}$$

$$\mathcal{J}(u,z,r) = \fint_{\mathcal{D}(z,r)} d\xi d\eta\,[u(\zeta)-u(z)] \tag{4.4}$$

where $z=x+iy, \zeta=\xi+i\eta$. Recall that $\mathcal{J}(u,z,r)\geq 0$ for any subharmonic function u.

Lemma 4.1. *Let $f(z)$ be analytic in $\mathcal{D}(z_0,R_0)$. Then for any $0<r_2<r_1<R_0-r_2$*

$$\nu_f(z_0,r_1-r_2) \leq 4\frac{r_1^2}{r_2^2}\mathcal{J}(\log|f|,z_0,r_1,r_2) \leq \nu_f(z_0,r_1+r_2)$$

Proof. Jensen's formula yields

$$\begin{aligned}
\mathcal{J}(\log|f|,z_0,r_1,r_2) &= \fint_{\mathcal{D}(z_0,r_1)} dx\,dy\left[\frac{2}{r_2^2}\int_0^{r_2} dr\Big(r\sum_{f(\zeta)=0,\zeta\in\mathcal{D}(z,r)}\log\Big(\frac{r}{|\zeta-z|}\Big)\Big)\right]\\
&\leq \sum_{f(\zeta)=0,\zeta\in\mathcal{D}(z_0,r_1+r_2)} \frac{1}{\pi r_1^2}\left[\frac{2}{r_2^2}\int_0^{r_2} dr\Big(r\int_{\mathcal{D}(\zeta,r)}\log\Big(\frac{r}{|z-\zeta|}\Big)dx\,dy\Big)\right]\\
&= \frac{1}{4}\frac{r_2^2}{r_1^2}\nu_f(z_0,r_1+r_2),
\end{aligned}$$

which proves the upper estimate for $\mathcal{J}(\log|f|,z_0,r_1,r_2)$. The proof of the lower estimate is similar. $\square$

Corollary 4.2. *Let f be analytic in $\mathcal{D}(z_0,R_0)$, $0<r_2<r_1<R_0-r_2$. Assume that f has no zeros in the annulus $\mathcal{A}=\big\{r_1-r_2\leq|z-z_0|\leq r_1+r_2\big\}$. Then*

$$\nu_f(z_0,r_1) = 4\,\frac{r_1^2}{r_2^2}\,\mathcal{J}\big(\log|f|,z_0,r_1,r_2\big)$$

Corollary 4.3. *Let $f(z), g(z)$ be analytic in $\mathcal{D}(z_0, R_0)$. Assume that for some $0 < r_2 < r_1 < R_0 - r_2$*

$$|\mathcal{J}(\log|f|, z_0, r_1, r_2) - \mathcal{J}(\log|g|, z_0, r_1, r_2)| < \frac{r_2^2}{4r_1^2}$$

Then $\nu_f(z_0, r_1 - r_2) \le \nu_g(z_0, r_1 + r_2)$, $\nu_g(z_0, r_1 - r_2) \le \nu_f(z_0, r_1 + r_2)$.

We shall also need a simple generalization of these estimates to averages over general domains. More precisely, set

$$\nu_f(\mathcal{D}) = \#\{z \in \mathcal{D} : f(z) = 0\} \tag{4.5}$$

$$\mathcal{J}(u, \mathcal{D}, r_2) = \fint_{\mathcal{D}} dx\, dy \fint_{\mathcal{D}(z,r_2)} d\xi d\eta\, [u(\zeta) - u(z)].$$

Given a domain $\mathcal{D}$ and $r > 0$, set $\mathcal{D}(r) = \{z : \mathrm{dist}(z, \mathcal{D}) < r\}$. Let $f(z)$ be analytic in $\mathcal{D}(R)$. Then for any $0 < r_2 < r_1 < R - r_2$

$$\nu_f(\mathcal{D}(r_1 - r_2)) \le 2\frac{\mathrm{mes}\,(\mathcal{D})}{\pi r_2^2}\mathcal{J}(\log|f|, \mathcal{D}(r_1), r_2) \le \nu_f(\mathcal{D}(r_1 + r_2)) \tag{4.6}$$

Let $\mathcal{A}_{R_1,R_2} := \{z \in \mathbb{C} \,:\, R_1 < |z| < R_2\}$.

Lemma 4.4.

$$\begin{aligned} &N^{-1}\mathcal{J}\big(\log\big|f_N(\cdot,\omega,E)\big|, \mathcal{A}_{R_1,R_2}, r_2\big) = \\ &4(R_2^2 - R_1^2)^{-1} r_2^{-2} \int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r\, dr \int_0^1 dy \big[L_N(\xi(\rho,r,y),\omega,E) - L_N(\xi(\rho),\omega,E)\big] \end{aligned} \tag{4.7}$$

where $\xi(\rho, r, y) = \log|\rho + re(y)|$, $\xi(\rho) = \log\rho$.

Proof. Due to the definition of $\mathcal{J}(u, \mathcal{D}, r_2)$ one has

$$\begin{aligned} &N^{-1}\mathcal{J}(\log|f_N(\cdot,\omega,E)|, \mathcal{A}_{R_1,R_2}, r_2) \\ &= \frac{4\pi N^{-1}}{|\mathcal{A}_{R_1,R_2}| r_2^2} \int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r\, dr \bigg\{ \int_0^1 dx \int_0^1 dy \big[\log|f_N(\rho e(x) + re(y),\omega,E)| - \log|f_N(\rho e(x),\omega,E)|\big] \bigg\} \\ &= \frac{4\pi N^{-1}}{|\mathcal{A}_{R_1,R_2}| r_2^2} \int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r\, dr \bigg\{ \int_0^1 dx \int_0^1 dy \big[\log|f_N(|\rho + re(y)|e(x),\omega,E)| - \log|f_N(\rho e(x),\omega,E)|\big] \bigg\} \\ &= 4(R_2^2 - R_1^2)^{-1} r_2^{-2} \int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r\, dr \int_0^1 dy \big[L_N(\xi(\rho,r,y),\omega,E) - L_N(\xi(\rho),\omega,E)\big] \end{aligned}$$

as claimed. □

Set

$$\mathcal{M}_N(\omega, E, R_1, R_2) := N^{-1}\#\big\{z \in \mathcal{A}_{R_1,R_2} : f_N(z,\omega,E) = 0\big\} \tag{4.8}$$

Remark 4.5. *Recall that $\log|f_N(z,\omega,E)| \le C(V)N$. Corollary 4.2 and the previous lemma therefore imply that*

$$\mathcal{M}_N(\omega, E, R_1, R_2) \le C(V)$$

for any N and R_1, R_2. Furthermore, if $V(e(x))$ is a trigonometric polynomial of degree k_0 then

$$V(z) = z^{-k_0}P(z)$$

where $P(z)$ is a polynomial of degree $2k_0$. Hence, with ω and E being fixed one has

$$z^{Nk_0} f_N(z, \omega, E) = F_N(z)$$

where $F_N(z)$ is a polynomial of degree $2Nk_0$. Therefore, in this case

$$\mathcal{M}_N(\omega, E, R_1, R_2) \le 2k_0$$

which will be crucial for the $2k_0$ bound at the end of Theorem 1.3.

The following lemma allows us to compare these averages for different scales. Later, this will be the crucial device that prevents “pre-gaps” from collapsing at subsequent stages of the iteration.

Lemma 4.6. *Assume $\gamma = L(\omega, E) > 0$ and fix some small $0 < \sigma \ll 1$. There exist $N_0 = N_0(V, \omega, \gamma, \sigma)$, $\rho^{(0)} = \rho^{(0)}(V, \omega, \gamma) > 0$ such that for any $n > N_0$, $N > \exp(\gamma\, n^\sigma)$, $1 - \rho^{(0)} < R_1 < R_2 < 1 + \rho^{(0)}$ one has*

$$\begin{aligned} &\mathcal{M}_N(\omega, E, R_1 + r_2, R_2 - r_2) \le \mathcal{M}_n(\omega, E, R_1 - r_2, R_2 + r_2) + n^{-1/4} \\ &\mathcal{M}_n(\omega, E, R_1 + r_2, R_2 - r_2) \le \mathcal{M}_N(\omega, E, R_1 - r_2, R_2 + r_2) + n^{-1/4} \end{aligned} \tag{4.9}$$

where $r_2 = n^{-1/4}(R_2 - R_1)$ and provided $r_2 > \exp(-\gamma\, n^\sigma/100)$.

Proof. Recall that due to avalanche principle expansion one has

$$\begin{aligned} \Big| \log \frac{\big\|M_n\big(e(x+n\omega+iy),\omega,E\big)\big\| \; \big\|M_n\big(e(x+iy),\omega,E\big)\big\|}{\big\|M_{2n}\big(e(x+iy),\omega,E\big)\big\|} - \\ \log \frac{\big\|M_\ell\big(e(x+n\omega+iy)\big)\big\| \; \big\|M_\ell\big(e(x+(n-\ell)\omega+iy),\omega,E\big)\big\|}{\big\|M_{2\ell}\big(e(x+(n-\ell)\omega+iy)\big\|} \Big| \le \exp\big(-\gamma_1\, n^{1/2}\big) \end{aligned}$$

for any $|y| < \rho_0/2$, $x \in \mathbb{T} \setminus \mathcal{B}_y$, mes $\mathcal{B}_y < \exp\big(-\gamma_1\, n^{1/2}\big)$ where $\ell = \big[n^{1/2}\big]$, $\gamma_k = L(\omega, E)/2^k$.

That implies in particular

$$\begin{aligned} &L_n(y, \omega, E) - L_{2n}(y, \omega, E) = \\ &\frac{\ell}{n}(L_\ell(y, \omega, E) - L_{2\ell}(y, \omega, E)) + O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big) \end{aligned} \tag{4.10}$$

Let $\xi(\rho) = \log\rho$, $\xi(\rho, r, y) = \log|\rho + re(y)|$, $R_1 < \rho < R_2$, $0 < r < r_2$, $0 \le y \le 1$, as in Lemma 4.4. Then, by Lemma 2.10

$$\big|L_{j\ell}\big(\xi(\rho, r, y), \omega, E\big) - L_{j\ell}\big(\xi(\rho), \omega, E\big)\big| \le C R_1^{-1} r \qquad j = 1, 2 \tag{4.11}$$

Recall that for any $N > \exp(\gamma_1\, n^\sigma)$ one has

$$\big|L_N(y, \omega, E) - 2L_{2n}(y, \omega, E) + L_n(y, \omega, E)\big| < \exp\big(-\gamma_2 n^\sigma\big)\ ,$$

see [GolSch1]. Hence, due to (4.6) and Lemma 4.4

$$\mathcal{M}_N(\omega, E, R_1 + r_2, R_2 - r_2) \le \frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2 N}\, \mathcal{J}\Big(\log\big|f_N(\cdot, \omega, E)\big|, \mathcal{A}_{R_1,R_2}, r_2\Big)$$

$$= \frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2}\, \mathcal{J}\Big(n^{-1}[\log\big|f_{2n}(\cdot, \omega, E)\big| - \log\big|f_n(\cdot, \omega, E)\big|], \mathcal{A}_{R_1,R_2}, r_2\Big) \tag{4.12}$$

$$+ O\big((R_2 - R_1) r_2^{-2} \exp(-\gamma_2 n^\sigma)\big) \tag{4.13}$$

Next, we rewrite the Jensen average in (4.12) using Lemma 4.4

$$\mathcal{J}\Big(n^{-1}[\log\big|f_{2n}(\cdot, \omega, E)\big| - \log\big|f_n(\cdot, \omega, E)\big|], \mathcal{A}_{R_1,R_2}, r_2\Big)$$

$$= 2\mathcal{J}\Big(\frac{1}{2n}\log\big|f_{2n}(\cdot, \omega, E)\big| - \frac{1}{n}\log\big|f_n(\cdot, \omega, E)\big|, \mathcal{A}_{R_1,R_2}, r_2\Big) \tag{4.14}$$

$$+ \mathcal{J}\Big(n^{-1}\log\big|f_n(\cdot, \omega, E)\big|, \mathcal{A}_{R_1,R_2}, r_2\Big) \tag{4.15}$$

Inserting (4.15) into (4.12) leads to the main term on the right-hand side of (4.9). It is bounded above by $\mathcal{M}_n(\omega, E, R_1 - r_2, R_2 + r_2)$ in view of (4.6). It remains to bound the error term (4.14). We introduce the short-hand notation

$$\begin{aligned} &\mathcal{S}[L_n\big(\xi(\rho, r, y), \omega, E\big) - L_n\big(\xi(\rho), \omega, E\big)] \\ &= \frac{4\pi}{(R_2^2 - R_1^2) r_2^2} \int_{R_1}^{R_2} \rho\, d\rho \int_0^{r_2} r dr \int_0^1 dy \big[L_n\big(\xi(\rho, r, y), \omega, E\big) - L_n\big(\xi(\rho), \omega, E\big)\big] \end{aligned}$$

Hence, the Jensen-average in (4.14) equals, see (4.10),

$$
\begin{aligned}
&\mathcal{S}[L_{2n}\big(\xi(\rho,r,y),\omega,E\big) - L_n\big(\xi(\rho,r,y),\omega,E\big)] - \mathcal{S}[L_{2n}\big(\xi(\rho),\omega,E\big) - L_n\big(\xi(\rho),\omega,E\big)] \\
&= \frac{\ell}{n}\mathcal{S}[L_{2\ell}\big(\xi(\rho,r,y),\omega,E\big) - L_\ell\big(\xi(\rho,r,y),\omega,E\big)] - \frac{\ell}{n}\mathcal{S}[L_{2\ell}\big(\xi(\rho),\omega,E\big) - L_\ell\big(\xi(\rho),\omega,E\big)] \\
&\quad + O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)
\end{aligned}
$$

By the Lipschitz bound (4.11), we can further estimate the absolute value here by

$$
\begin{aligned}
&\lesssim \Big|\frac{\ell}{n}\mathcal{S}[L_{2\ell}\big(\xi(\rho,r,y),\omega,E\big) - L_{2\ell}\big(\xi(\rho),\omega,E\big)]\Big| + \Big|\frac{\ell}{n}\mathcal{S}[L_\ell\big(\xi(\rho,r,y),\omega,E\big) - L_\ell\big(\xi(\rho),\omega,E\big)]\Big| \\
&\quad + O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big) \\
&\lesssim n^{-1/2} r_2 + O\Big(\exp\big(-\gamma_2 n^{1/2}\big)\Big)
\end{aligned}
$$

So the total error is the sum of this term times $\frac{4|\mathcal{A}_{R_1,R_2}|}{r_2^2}$ plus the error in (4.13). In view of our assumptions on r_2 the lemma is proved. ☐

5. Eliminating resonances via resultants

In this section we describe the mechanism behind the process of eliminating "bad" frequencies ω, which is fundamental to everything we do. "Bad" here refers to those ω which produce too many too close resonances. More precisely, we will need to ensure that the zeros of

$$
f_{\ell_1}(\cdot,\omega,E) \quad \text{and} \quad f_{\ell_2}(\cdot e(t\omega),\omega,E)
$$

do not come too close. This requires the elimination of a set of energies of small measure and not too large complexity. The elimination method is based on the natural idea that ω and $t\omega$ act almost as independent variables (due to ω being the slow and $t\omega$ being the fast variable). On a technical level this will be accomplished via a Cartan estimate on the *resultant* of two polynomials (which themselves come from the Weierstrass preparation theorem applied to f_{ℓ_1}, f_{ℓ_2}). For those properties of the resultant which we need here, we refer the reader to Appendix A. We would like to draw the reader's attention to the fact that all sets removed in this section are very small in terms of Hausdorff dimension. Indeed, the complexity of the bad sets is always less than their measure raised to an arbitrarily small negative number (at least for f_N with N large). This is one reason why we are able to eventually remove sets of Hausdorff dimension zero. Another reason has to do with the second, an completely different, elimination method used in this paper. It is designed to remove triple resonances and is based on the implicit function theorem rather than resultants, see Section 14. It should be emphasized, though, that Section 14 must come after this section in the sense that the methods there can only possibly work after we have removed the frequencies (as well as energies) specified in this section. This is simply due to the fact that the implicit function theorem requires a non-degeneracy condition which can be guaranteed only via the results of this section.

Lemma 5.1. *Let $f(z;w) = z^k + a_{k-1}(w)z^{k-1} + \cdots + a_0(w)$, $g(z;w) = z^m + b_{m-1}(w)z^{m-1} + \cdots + b_0(w)$ be polynomials whose coefficients $a_i(w)$, $b_j(w)$ are analytic functions defined in a domain $G \subset \mathbb{C}^d$. Then $\mathrm{Res}(f(\cdot,w),g(\cdot,w))$ is analytic in G.*

Our goal here is to separate the zeros of two analytic functions using the resultants by means of shifts in the argument, see Section 7 of [GolSch2], in particular Lemma 7.4. This can be reduced to the same question for polynomials due to the Weierstrass preparation theorem. Here is a simple observation regarding the resultant of a polynomial and a shifted version of another polynomial.

Lemma 5.2. *Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$, $g(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_0$ be polynomials. Then*

$$
\mathrm{Res}\big(f(\cdot + w), g(\cdot)\big) = (-w)^n + c_{n-1}w^{n-1} + \cdots + c_0 \tag{5.1}
$$

where $n = km$, and $c_0, c_1 \cdots$ are polynomials in the a_i, b_j.

Proof. Let ζ_j, $1 \le j \le k$ (resp. $\eta_i, 1 \le i \le m$) be the zeros of $f(\cdot)$ (resp. $g(\cdot)$). The zeros of $f(\cdot + w)$ are $\zeta_j - w$, $1 \le j \le k$. Hence

$$\operatorname{Res}\big(f(\cdot + w), g(\cdot)\big) = \prod_{i,j} (\zeta_j - w - \eta_i) \tag{5.2}$$

and (5.1) follows. □

The following lemma gives some information on the coefficients in (5.1).

Lemma 5.3. *Let $P_s(z, \underline{w}) = z^{k_s} + a_{s,k_s-1}(\underline{w}) z^{k_s-1} + \cdots + a_{s,0}(\underline{w})$, $z \in \mathbb{C}$, where $a_{s,j}(\underline{w})$ are analytic functions defined in some polydisk $\mathcal{P} = \prod_i \mathcal{D}(w_{i,0}, r)$, $\underline{w} = (w_1, \ldots, w_d) \in \mathbb{C}^d$, $\underline{w}_0 = (w_{1,0}, \ldots, w_{d,0}) \in \mathbb{C}^d$, $s = 1, 2$. Set $\chi(\eta, \underline{w}) = \operatorname{Res}\big(P_1(\cdot, \underline{w}), P_2(\cdot + \eta, \underline{w})\big)$, $\eta \in \mathbb{C}$, $\underline{w} \in \mathcal{P}$. Then*

$$\chi(\eta, \underline{w}) = (-\eta)^k + b_{k-1}(\underline{w}) \eta^{k-1} + \cdots + b_0(\underline{w}) \tag{5.3}$$

where $k = k_1 k_2$, $b_j(\underline{w})$ are analytic in $\mathcal{P}$, $j = 0, 1, \ldots, k-1$. Moreover, if the zeros of $P_i(\cdot, \underline{w})$ belong to the same disk $D(z_0, r_0)$, $i = 1, 2$ then for all $0 \le j \le k-1$,

$$\big|b_j(\underline{w})\big| \le \binom{k}{k-j} (2r_0)^{k-j} \le (2r_0 k)^{k-j} \tag{5.4}$$

Proof. The relation (5.3) with some coefficients $b_j(\underline{w})$ follows from Lemma 5.2 whereas the bound (5.4) follows from the expansion (5.2). By Lemma 5.2, $\chi(\eta, \underline{w})$ is analytic in $\mathbb{C} \times \mathcal{P}$. Therefore $b_j(\underline{w}) = (j!)^{-1} (\partial_\eta)^j \chi(\eta, \underline{w}) \Big|_{\eta = 0}$ are analytic $j = 0, 1, \ldots, k-1$. □

The following lemma allows for the separation of the zeros of one polynomial from those of a shifted version of another polynomial. This will be the main mechanism for eliminating certain "bad" rotation numbers ω. The logic is as follows: due to the basic definition of the resultant,

$$|\chi(\eta, \underline{w})| = \prod_{i,j} \big|\zeta_{i,1}(\underline{w}) - \zeta_{j,2}(\eta, \underline{w})\big|$$

where $\zeta_{i,1}(\underline{w})$, $\zeta_{j,2}(\eta, \underline{w})$ are the zeros of $P_1(\cdot, \underline{w})$ and $P_2(\cdot + \eta, \underline{w})$, respectively. Therefore, if $\big|\zeta_{i,1}(\underline{w}) - \zeta_{j,2}(\underline{w})\big| < \exp(-kH)$ for one choice of i, then $\big|\chi(\eta, \underline{w})\big| < \exp(-kH)$. This simple fact allows one to separate the zeros $\zeta_{i,1}(\underline{w})$ from the zeros $\zeta_{j,2}(\underline{w})$ provided $\underline{w}$ falls outside of a set whose measure and complexity is controlled by Cartan's estimate. More specifically, we obtain this separation by means of a shift by $t\omega_1$ in the z-slot:

Lemma 5.4. *Let $P_s(z, \underline{w})$ be polynomials in z as in Lemma 5.3, $s = 1, 2$. In particular, $\underline{w} \in \mathcal{P}$ where $\mathcal{P}$ is a polydisk of some given radius $r > 0$. Assume that $k_s > 0$, $s = 1, 2$ and set $k = k_1 k_2$. Suppose that for any $\underline{w} \in \mathcal{P}$ the zeros of $P_s(\cdot, \underline{w})$ belong to the same disk $\mathcal{D}(z_0, r_0)$, $r_0 \ll 1$, $s = 1, 2$. Let $t > 16k\, r_0\, r^{-1}$. Given $H \gg 1$ there exists a set*

$$\mathcal{B}_{H,t} \subset \widetilde{\mathcal{P}} := \mathcal{D}(w_{1,0}, 8kr_0/t) \times \prod_{j=2}^{d} \mathcal{D}(w_{j,0}, r/2)$$

such that $S_{\underline{w}_0, (16kr_0t^{-1}, r, \ldots, r)}(\mathcal{B}_{H,t}) \in \operatorname{Car}_d(H^{1/d}, K)$, $K = CHk$ and for any $\underline{w} \in \widetilde{\mathcal{P}} \setminus \mathcal{B}_{H,t}$ one has

$$\operatorname{dist}\big(\{\text{zeros of } P_1(\cdot, \underline{w})\}, \{\text{zeros of } P_2\big(\cdot + t(w_1 - w_{1,0}), \underline{w}\big)\}\big) \ge e^{-CHk} \tag{5.5}$$

Proof. Define $\chi(\eta, \underline{w})$ as in Lemma 5.3. Note that for any $\underline{w} \in \mathcal{P}$ one has

$$|\chi(\eta, \underline{w})| \ge |\eta|^k \big[1 - \sum_{j=1}^{\infty} \big(\frac{2r_0 k}{|\eta|}\big)^j\big] \ge \frac{1}{2} |\eta|^k$$

provided $|\eta| \geq 8r_0\, k$. Furthermore, for any $\underline{w} \in \mathcal{P}$,

$$|\chi(\eta, \underline{w})| \leq |\eta|^k \big[1 + \sum_{j=1}^{\infty} \big(\frac{2r_0 k}{|\eta|}\big)^j\big] \leq 2|\eta|^k.$$

provided $|\eta| \geq 8r_0\, k$. Hence, by the maximum principle,

$$\sup\big\{\big|\chi(\eta, \underline{w})\big| : |\eta| \leq 16r_0 k\big\} \leq 2(16kr_0)^k \ .$$

Set

$$f(\underline{w}) = \chi\big(t(w_1 - w_{1,0}), (w_1, w_2, \ldots, w_d)\big), \quad w_1 \in \mathcal{D}(w_{1,0}, 16kr_0/t), \quad (w_2, \ldots, w_d) \in \prod_{j=2}^{d} \mathcal{D}(w_{j,0}, r).$$

This function is well-defined because $16kr_0/t < r$ by our lower bound on t. By the preceding,

$$\sup_{\underline{w}} |f(\underline{w})| \leq 2(16kr_0)^k, \quad \big|f\big(w_{1,0} + 8kr_0/t, w_{2,0}, \ldots, w_{d,0}\big)\big| > \frac{1}{2}(8kr_0)^k.$$

We can therefore apply Lemma 2.4 to

$$\phi = f \circ S^{-1}_{\underline{w}_0,(16kr_0/t, r, \ldots, r)} \ \text{ with } \ M = \log 2 + k\log(16kr_0), \quad m = -\log 2 + k\log(8kr_0)$$

on a polydisk of unit size. Thus, given $H \gg 1$ there exists $\mathcal{B}^{(1)}_{H,t} \subset \mathcal{P}$ such that

$$S_{\underline{w}_0,(16kr_0t^{-1}, r, \ldots, r)}\big(\mathcal{B}^{(1)}_{H,t}\big) \in \mathrm{Car}_d\big(H^{1/d}, K\big), \quad K = CkH,$$

and such that for any

$$(w_1, \ldots, w_d) \in \mathcal{D}(w_{1,0}, 8kr_0/t) \times \prod_{j=2}^{d} \mathcal{D}(w_{j,0}, r/2) \setminus \mathcal{B}^{(1)}_{H,t}$$

one has $|f(\underline{w})| > e^{-CHk}$. Recall that due to basic properties of the resultant

$$|f(w)| = \prod_{i,j} |\zeta_{i,1}(\underline{w}) - \zeta_{j,2}(\underline{w})|$$

where $\zeta_{i,1}(\underline{w})$, $\zeta_{j,2}(\underline{w})$ are the zeros of $P_1(\cdot, \underline{w})$, and $P_2(\cdot + t(w_1 - w_{1,0}), \underline{w})$, respectively. Since $r_0 \ll 1$, this implies (5.5), and we are done. ☐

Lemma 5.4 of course applies to polynomials $P_s(z)$ that do not depend on $\underline{w}$ at all. This example is important, and explains why quantities like K have the stated form.

This method of elimination applies to the Dirichlet determinants $f_{\ell_1}(\cdot, \omega, E)$ and $f_{\ell_2}(\cdot e(t\omega), \omega, E)$. We now state a result in this direction. We shall use the following notation

$$\mathcal{Z}(f, \Omega) = \big\{z \in \Omega : f(z) = 0\big\}$$

and

$$\mathcal{Z}(f, z_0, r_0) = \mathcal{Z}(f, \mathcal{D}(z_0, r_0))$$

Proposition 5.5. *Let V be analytic on $\mathcal{A}_{\rho_0}$ and real-valued on $\mathbb{T}$. Assume that $L(\omega, E) > \gamma > 0$ for all[2] ω, E. Fix $a > 1$, and $1 \gg c > 0$ as well as a bounded set $\mathcal{S} \subset \mathbb{C}$. There exists $\ell_0 = \ell_0(V, \rho_0, a, c, \gamma, \mathcal{S})$, such that for any $\ell_1 \geq \ell_2 \geq \ell_0$ the following holds: Given $t > \exp\big(\big(\log \ell_1\big)^{C_0}\big)$, $H \geq 1$, there exists a set $\Omega_{\ell_1,\ell_2,t,H} \subset \mathbb{T}$, with*

$$\begin{aligned} \mathrm{mes}\,\big(\Omega_{\ell_1,\ell_2,t,H}\big) &< \exp\big((\log \ell_1)^{C_1}\big) e^{-\sqrt{H}} \\ \mathrm{compl}\big(\Omega_{\ell_1,\ell_2,t,H}\big) &< t\exp\big((\log \ell_1)^{C_1}\big) H \end{aligned} \tag{5.6}$$

[2]One can localize here to intervals of ω and E.

such that for any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{\ell_1,\ell_2,t,H}$ there exists a set $\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ with

$$\mathrm{mes}\,\big(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}\big) < t\exp\big((\log \ell_1)^{C_1}\big)\, e^{-\sqrt{H}}$$
$$\mathrm{compl}\big(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}\big) < t\exp\big((\log \ell_1)^{C_1}\big) H$$

such that for any $E \in \mathcal{S} \setminus \mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ one has

$$\mathrm{dist}\big(\mathcal{Z}\big(f_{\ell_1}(\cdot,\omega,E),\mathcal{A}_{\rho_0}\big), \mathcal{Z}\big(f_{\ell_2}(\cdot e(t\omega),\omega,E),\mathcal{A}_{\rho_0}\big)\big) > e^{-H(\log \ell_1)^{C_2}} \tag{5.7}$$

$$\mathrm{dist}\big(\mathrm{spec}\big(H_{\ell_1}(e(x_0),\omega)\big) \setminus \mathcal{E}_{\ell_1,\ell_2,t,H,\omega}\ ,\ \mathrm{spec}\big(H_{\ell_2}(e(x_0+t\omega),\omega)\big)\big) \ge e^{-H(\log \ell_1)^{3C_2}}. \tag{5.8}$$

Here C_j are also allowed to depend on $\mathcal{S}$.

Proof. Fix some choice of $z_0 \in \mathcal{A}_{\rho_0}$, $E_0 \in \mathcal{S}$, and $\omega_0 \in \mathbb{T}_{c,a}$. By the Weierstrass preparation theorem of the previous section we can write

$$f_{\ell_1}(e(z),\omega,E) = P_1(z,\omega,E) g_1(z,\omega,E)$$
$$f_{\ell_2}(e(z+t\omega_0),\omega,E) = P_2(z,\omega,E) g_2(z,\omega,E)$$

for all

$$(z,\omega,E) \in \mathcal{P}_0 := \mathcal{D}(z_0,r_0) \times \mathcal{D}(\omega_0,r) \times \mathcal{D}(E_0,r)$$

where[3] $r_0 \asymp \ell_1^{-1}$, $r \asymp \exp(-(\log \ell_1)^{C_0})$, and g_j does not vanish on $\mathcal{P}_0$. Moreover, each $P_j(\cdot,\omega,E)$ is a polynomial of degree $k_j \lesssim (\log \ell_j)^{C_0}$ for all $(\omega,E) \in \mathcal{D}(\omega_j,r) \times \mathcal{D}(E_i,r)$ and all its zeros belong to $\mathcal{D}(z_0,r_0)$. Apply Lemma 5.4, with $t > (\log \ell_1)^{C_1} > 16k_1k_2r_0r^{-1}$, to the polynomials

$$P_1(\cdot,\omega,E), \qquad P_2(\cdot + t(\omega-\omega_0),\omega,E)$$

Thus, for any $H \ge 1$ there exists $\mathcal{B}_{H,t} \subset \mathcal{D}(\omega_0, 8kr_0/t) \times \mathcal{D}(E_0,r)$ with

$$\Big\{\big(t(\omega-\omega_0)/(16kr_0), (E-E_0)/r\big)\ :\ (\omega,E) \in \mathcal{B}_{H,t}\Big\} \in \mathrm{Car}_2(H^{1/2},K), \quad K = CHk,$$

so that for any $(\omega,E) \in \mathcal{D}(\omega_0,8kr_0/t) \times \mathcal{D}(E_0,r/2) \setminus \mathcal{B}_{H,t}$ one has

$$\mathrm{dist}\big(\big\{\text{zeros of } P_1(\cdot,\omega,E)\big\}, \big\{\text{zeros of } P_2\big(\cdot + t(w_1 - w_{1,0}),\omega,E\big)\big\}\big) \ge e^{-CHk} \tag{5.9}$$

By definition of P_1, P_2, (5.9) this implies that

$$\mathrm{dist}\big(\mathcal{Z}\big(f_{\ell_1}(\cdot,\omega,E),z_0,r_0\big), \mathcal{Z}\big(f_{\ell_2}(\cdot e(t\omega),\omega,E),z_0,r_0\big)\big) > e^{-H(\log \ell_1)^{C_1}}$$

Now let z_0, ω_0, E_0 run over a net

$$\mathcal{N} = \{(z_j,\omega_j,E_j)\}_{j=1}^J \subset \mathcal{A}_{\rho_0} \times \mathbb{T}_{c,a} \times \mathcal{S}$$

so that each point in $\mathcal{A}_{\rho_0} \times \mathbb{T}_{c,a} \times \mathcal{S}$ comes $(r_0, kr_0/t, r)$–close to one of the points in $\mathcal{N}$ and no two points in $\mathcal{N}$ are closer than this distance. Denoting by $\mathcal{B}_{H,t}(j)$ the bad set constructed above for each point in $\mathcal{N}$, there exist

$$\Omega_j \in \omega_j + t^{-1}kr_0\,\mathrm{Car}_1(\sqrt{H},K)$$

so that for each

$$z \in \mathcal{D}(\omega_j, kr_0/t) \setminus \Omega_j$$

the z-slice $\mathcal{E}_{j,z} := \mathcal{B}_{H,t}(j)|_z$ belongs to $E_j + r\,\mathrm{Car}_1(\sqrt{H},K)$. Now define

$$\Omega_{\ell_1,\ell_2,t,H} := \bigcup_j \Omega_j$$

By construction, $\Omega_{\ell_1,\ell_2,t,H}$ satisfies (5.6). Moreover, for each $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{\ell_1,\ell_2,t,H}$ define

$$\mathcal{E}_{\ell_1,\ell_2,t,H,\omega} := \bigcup_j \mathcal{E}_{j,z}$$

[3]We remind the reader that $\asymp$ means proportional. The constant of proportionality here is allowed to depend on $V, \rho_0, a, c, \gamma, \mathcal{S}$.

Then

$$\mathrm{mes}\,(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}) \le t\exp((\log\ell_1)^{C_1})e^{-\sqrt{H}}, \quad \mathrm{compl}(\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}) \le \exp((\log\ell_1)^{C_1})tH^2$$

as desired. If (5.7) failed, then there would have to be $z_1, z_2 \in \mathcal{A}_{\rho_0}$ and $\omega_0 \in \mathbb{T}_{c,a} \setminus \Omega_{\ell_1,\ell_2,t,H}$ and $E_0 \in \mathcal{S} \setminus \mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ such that

$$f_{\ell_1}(z_1,\omega_0,E_0) = f_{\ell_2}(z_2 e(t\omega_0),\omega_0,E_0) = 0, \quad |z_1 - z_2| < e^{-H(\log\ell_1)^{C_2}}$$

Then $(z_1,\omega_0,E_0) \in \mathcal{D}(z_j,r_0) \times \mathcal{D}(\omega_j,kr_0/t) \times \mathcal{D}(E_0,r)$ for some choice of j. By construction, $\omega_0 \in \mathcal{D}(\omega_j,kr_0/t) \setminus \Omega_j$ and $E_0 \in \mathcal{D}(E_0,r) \setminus \mathcal{E}_{j,z}$ which implies that

$$|z_1 - z_2| \ge e^{-CHk},$$

see (5.9). This is a contradiction and we are done with (5.7). For (5.8), assume that $f_{\ell_1}(z_1,\omega,E_1) = 0$, $f_{\ell_2}\big(z_1 e(t\omega),\omega,E_2\big) = 0$ for arbitrary $z_1 = e(x_0)$, and

$$|E_1 - E_2| < e^{-H(\log\ell_1)^{3C_2}}, \qquad E_1 \in [-C,C] \setminus \mathcal{E}_{\ell_1,\ell_2,t,H,\omega},\ \omega \in \mathbb{T} \setminus \Omega_{\ell_1,\ell_2,t,H}\,. \tag{5.10}$$

Then, by Corollary 2.14,

$$\big|f_{\ell_2}(z_1 e(t\omega),\omega,E_1)\big| \lesssim |E_1 - E_2|\exp\big(\ell_2 L(\omega,E_1) + (\log\ell_2)^B\big) < \exp\big(\ell_2 L(\omega,E_1) - H(\log\ell_1)^{2C_2}\big)$$

By our choice of E_1, there exists z_2 so that $|z_2 - z_1| < \exp(-100H(\log\ell_1)^{C_2})$, for which

$$f_{\ell_2}\big(z_2 e(t\omega),\omega,E_1\big) = 0,$$

see Corollary 2.18. But this would contradict (5.7) and we are done. □

6. Localized eigenfunctions in finite volume

In this section we apply the results of the previous section to the study of the eigenfunctions of the Hamiltonian restricted to intervals on the integer lattice. More precisely, we shall obtain a finite-volume version of Anderson localization (albeit, at the expense of removing a small set of energies). This section corresponds to Section 9 of [GolSch2].

Lemma 6.1. *Let $\omega \in \mathbb{T}_{c,a}$, $E_0 \in \mathbb{R}$, $L(\omega,E_0) > \gamma > 0$, and $N \ge N_0(V,\rho_0,a,c,\gamma,E_0)$. Furthermore, assume that*

$$\log\big|f_N(z_0,\omega,E_0)\big| > NL(\omega,E_0) - K/2 \tag{6.1}$$

for some $z_0 = e(x_0)$, $x_0 \in \mathbb{T}$, $K > (\log N)^{C_0}$. Then

$$\big|\mathcal{G}_{[1,N]}(z_0,\omega,E)(j,k)\big| \le \exp\big(-\gamma(k-j) + K\big) \tag{6.2}$$

$$\big\|\mathcal{G}_{[1,N]}(z_0,\omega,E)\big\| \le \exp(K) \tag{6.3}$$

where $\mathcal{G}_{[1,N]}(z_0,\omega,E_0) = \big(H(z_0,\omega) - E_0\big)^{-1}$ is the Green function, $\gamma = L(\omega,E_0)$, $1 \le j \le k \le N$.

Proof. By Cramer's rule and the uniform upper bound of Proposition 2.12 as well as the rate of convergence estimate (4.10),

$$\begin{aligned}\big|\mathcal{G}_{[1,N]}(z_0,\omega,E)(j,k)\big| &= \big|f_{j-1}(z_0,\omega,E_0)\big| \cdot \big|f_{N-k}\big(z_0 e(k\omega),\omega,E_0\big)\big| \cdot \big|f_N(z_0,\omega,E_0)\big|^{-1} \\ &\le \big|f_N(z_0,\omega,E_0)\big|^{-1}\exp\big(NL(\omega,E_0) - (k-j)L(\omega,E_0) + (\log N)^C\big)\end{aligned} \tag{6.4}$$

Therefore, (6.2) follows from condition (6.1). The estimate (6.3) follows from (6.2) via Hilbert-Schmidt norms. □

Any solution of the equation

$$-\psi(n+1) - \psi(n-1) + v(n)\psi(n) = E\psi(n)\ , \quad n \in \mathbb{Z}\ , \tag{6.5}$$

obeys the relation ("Poisson formula")

$$\psi(m) = \mathcal{G}_{[a,b]}(E)(m,a-1)\psi(a-1) + \mathcal{G}_{[a,b]}(E)(m,b+1)\psi(b+1), \quad m \in [a,b]. \tag{6.6}$$

where $\mathcal{G}_{[a,b]}(E) = \big(H_{[a,b]} - E\big)^{-1}$ is the Green function, $H_{[a,b]}$ is the linear operator defined by (6.5) for $n \in [a,b]$ with zero boundary conditions. In particular, if ψ is a solution of equation (6.5), which satisfies a zero boundary condition at the left (right) edge, i.e.,

$$\psi(a-1) = 0 \quad (\text{resp. } \psi(b+1) = 0)\ ,$$

then

$$\psi(m) = \mathcal{G}_{[a,b]}(m, b+1)\psi(b+1)$$
$$\big(\text{resp. } \psi(m) = \mathcal{G}_{[a,b]}(m, a-1)\psi(a-1)\ \big)$$

The following lemma states that after removal of certain rotation numbers ω and energies E, but uniformly in $x \in \mathbb{T}$, only one choice of $n \in [1,N]$ can lead to a determinant $f_\ell(x+n\omega,\omega,E)$ with $\ell \asymp (\log n)^C$ which is not large. This relies on the elimination results, see (g) in Section 2, and is of crucial importance for all our work.

Lemma 6.2. *Fix $a > 1, c > 0$ and assume that $L(\omega,E) > \gamma > 0$ for all*[4] *ω, E. Given $N \geq N_0(V,\rho_0,\gamma,a,c)$ large, there exist a constant $B = B(V,\rho_0,\gamma,a,c)$ and a set $\Omega_N \subset \mathbb{T}$ with*

$$\operatorname{mes}(\Omega_N) < \exp\big(-(\log N)^B\big), \quad \operatorname{compl}(\Omega_N) < N^2,$$

such that for all $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ there is a set[5] *$\mathcal{E}_{N,\omega} \subset \mathbb{R}$,*

$$\operatorname{mes}(\mathcal{E}_{N,\omega}) < \exp\big(-(\log N)^B\big), \quad \operatorname{compl}(\mathcal{E}_{N,\omega}) < N^3,$$

with the following property: For any $x \in \mathbb{T}$ and any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, $E \in \mathbb{R} \setminus \mathcal{E}_{N,\omega}$ either

$$\log\big|f_\ell\big(e(x+n\omega),\omega,E\big)\big| > \ell L(\omega,E) - \sqrt{\ell} \tag{6.7}$$

for all

$$(\log N)^{20B} \leq \ell \leq 4(\log N)^{20B} \tag{6.8}$$

and all $1 \leq n \leq N$, or there exists $n_1 = n_1(x,\omega,E) \in [1,N]$ such that (6.7) holds for all $n \in [1,N] \setminus [n_1 - Q, n_1 + Q]$, with

$$Q = \big\lfloor \exp\big((\log\log N)^{B_1}\big)\big\rfloor, \quad B_1 = B_1(V,\rho_0,\gamma,a,c)$$

but not for $n = n_1$. Moreover, in this case

$$\big|f_{[1,n]}\big(e(x),\omega,E\big)\big| > \exp\big(nL(\omega,E) - (\log N)^{100B}\big) \tag{6.9}$$

for each $1 \leq n \leq n_1 - Q$ and

$$\big|f_{[n,N]}\big(e(x),\omega,E\big)\big| > \exp\big((N-n)L(\omega,E) - (\log N)^{100B}\big) \tag{6.10}$$

for each $n_1 + Q \leq n \leq N$.

Proof. Let $B\gamma \geq 2$ (below, we will need to make B large depending on a as well). With $a > 1$ and $c > 0$ fixed, we let $\Omega_{\ell_1,\ell_2,t,H}$ be as in Proposition 5.5 and define

$$\Omega_N := \bigcup \Omega_{\ell_1,\ell_2,t,H}$$

where $H = (\log N)^{3B}$ is fixed, and the union runs over ℓ_1, ℓ_2 as in (6.8), and $N > t > \exp\big((\log\log N)^{2C_0}\big)$ where C_0 is from Proposition 5.5 (thus, take $B_1 = 2C_0$). For any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ define

$$\mathcal{E}_{N,\omega} := \bigcup \mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$$

where $\mathcal{E}_{\ell_1,\ell_2,t,H,\omega}$ is from the proposition and the union is the same as before. Now fix $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, $E \in \mathbb{R} \setminus \mathcal{E}_{N,\omega}$ and suppose (6.7) fails somewhere, i.e.,

$$\log\big|f_{\ell_1}\big(e(x+n_1\omega),\omega,E\big)\big| < \ell_1 L(\omega,E) - \sqrt{\ell_1}$$

[4] This can of course be localized to intervals of ω and E.

[5] The sets $\Omega_N, \mathcal{E}_{N,\omega}$ also depend on V, ρ_0, γ, a, c but we omit these parameters from our notation.

for some $1 \le n_1 \le N$ and ℓ_1 as in (6.8). By Corollary 2.18 there exists z_1 with $|z_1 - e(x+n_1\omega)| < e^{-\ell_1^{\frac{1}{4}}}$ and

$$f_{\ell_1}(z_1, \omega, E) = 0$$

If

$$\log \bigl|f_{\ell_2}\bigl(e(x+n_2\omega), \omega, E\bigr)\bigr| < \ell_2 L(\omega, E) - \sqrt{\ell_2}$$

for some ℓ_2 as in (6.8) and $|n_2 - n_1| > \exp\bigl((\log\log N)^{B_1}\bigr)$, then for some z_2, and $t := n_1 - n_2$

$$f_{\ell_2}\bigl(z_2 e(t\omega), \omega, E\bigr) = 0$$

with

$$|z_1 - z_2| < e^{-(\log N)^{4B}}$$

However, by our choice of (ω, E)

$$|z_1 - z_2| > \exp\bigl(-CH(\log\log N)^{C_1}\bigr) = \exp\bigl(-C(\log N)^{3B}(\log\log N)^{C_1}\bigr)$$

which is a contradiction for $N \ge N_0$ large. Thus (6.7) holds for all ℓ as in (6.8), and any $1 \le n \le N$ such that $|n - n_1| > \exp\bigl((\log\log N)^{B_1}\bigr)$. This property allows one to apply the avalanche principle Proposition 2.8 to the determinants appearing in (6.9) and (6.10). It will suffice to consider the former with $n \ge (\log N)^{20B}$: in view of (6.7) the conditions of Proposition 2.8 hold if we choose the factor matrices A_j there to be of length as in (6.8). It yields that

$$\log \bigl|f_{[1,n]}\bigl(e(x), \omega, E\bigr)\bigr| \ge nL(\omega, E) - C\frac{n}{(\log N)^{5B}} > 0$$

provided N_0 is large. In fact, we can vary x here: note that by Corollary 2.15, if (6.7) holds at x, then also for all $z \in \mathcal{D}\bigl(e(x), e^{-\ell}\bigr)$, $\ell = (\log N)^{20B}$. Repeating the avalanche principle expansion for those z yields

$$f_{[1,n]}(z, \omega, E) \ne 0 \tag{6.11}$$

Now suppose

$$\log \bigl|f_{[1,n]}\bigl(e(x), \omega, E\bigr)\bigr| \le nL(\omega, E) - (\log N)^{100B}$$

By Corollary 2.18,

$$f_{[1,n]}(z, \omega, E) = 0$$

for some $|z - e(x)| < \exp\bigl(-(\log N)^{50B}\bigr)$ provided B is sufficiently large (depending on a). This contradicts (6.11) and we are done. □

Remark 6.3. *It follows from Corollary 2.15 that* (6.7) *is stable under perturbations of E by an amount $< e^{-C\ell}$. More precisely, if* (6.7) *holds for E, then*

$$\log \bigl|f_\ell\bigl(e(x+n\omega), \omega, E'\bigr)\bigr| > \ell L(E', \omega) - 2\sqrt{\ell}$$

for any E' with $|E' - E| < e^{-C\ell}$. Inspection of the previous proof now shows that (6.9) *and* (6.10) *are therefore also stable under such perturbations.*

The previous lemma yields the following finite volume version of Anderson localization.

Lemma 6.4. *Fix $a > 1, c > 0$, assume that $L(\omega, E) > \gamma > 0$ for all ω, E, and let Ω_N and $\mathcal{E}_{N,\omega}$ be as in the previous lemma with $N \ge N_0$. For any $x, \omega \in \mathbb{T}$, let $\bigl\{E_j^{(N)}(x,\omega)\bigr\}_{j=1}^N$ and $\bigl\{\psi_j^{(N)}(x,\omega,\cdot)\bigr\}_{j=1}^N$ denote the eigenvalues and normalized eigenvectors of $H_{[1,N]}(x,\omega)$, respectively. If $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ and for some j, $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$, then there exists a point $\nu_j^{(N)}(x,\omega) \in [1,N]$ (which we call the center of localization) so that for any $\exp\bigl((\log\log N)^{B_1}\bigr) \le Q \le N$ and with $\Lambda_Q := [1,N] \cap \bigl[\nu_j^{(N)}(x,\omega) - 2Q, \nu_j^{(N)}(x,\omega) + 2Q\bigr)$ one has*

(i) $\operatorname{dist}\bigl(E_j^{(N)}(x,\omega), \operatorname{spec}\bigl(H_{\Lambda_Q}(x,\omega)\bigr)\bigr) < e^{-\gamma Q/4}$

(ii) $\sum_{k \in [1,N]\setminus\Lambda_Q} \bigl|\psi_j^{(N)}(x,\omega;k)\bigr|^2 < e^{-Q\gamma}$, *where $\gamma > 0$ is a lower bound for the Lyapunov exponents.*

Here B, B_1 are the constants from the previous lemma.

Proof. Fix $N \ge N_0$, $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ and $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$. Let $n_1 = \nu_j^{(N)}(x,\omega)$ be such that

$$\big|\psi_j^{(N)}(x,\omega;n_1)\big| = \max_{1\le n\le N} \big|\psi_j^{(N)}(x,\omega;n)\big|$$

Fix an ℓ as in (6.8) and suppose that, with $E = E_j^{(N)}(x,\omega)$, and $\Lambda_0 := [1,N] \cap [n_1-\ell, n_1+\ell]$,

$$\log\big|f_{\Lambda_0}(x,\omega,E)\big| > |\Lambda_0| L(\omega,E) - \sqrt{\ell} \tag{6.12}$$

By Cramer's rule, see (2.17), this would then imply that

$$\big|G_{\Lambda_0}(x,\omega,E)(k,j)\big| < \exp\big(-\gamma\big|k-j\big| + C\sqrt{\ell}\big)$$

for all $k, j \in \Lambda_0$. But this contradicts the maximality of $\big|\psi_j^{(N)}(x,\omega;n_1)\big|$ due to (6.6) and ℓ being large. Hence (6.12) above fails, and we conclude from Lemma 6.2 that

$$\log\big|f_{\Lambda_1}(x,\omega,E)\big| > |\Lambda_1| L(\omega,E) - \sqrt{\ell}$$

for every $\Lambda_1 = [k-\ell, k+\ell] \cap [1,N]$ provided $|k-n_1| > \exp\big((\log\log N)^{B_1}\big)$. Since (6.12) fails, we conclude that $f_{\Lambda_0}(z_0,\omega,E) = 0$ for some z_0 with $|z_0 - e(x)| < e^{-\ell^{1/4}}$. By self-adjointness of $H_{\Lambda_0}(x,\omega,E)$ we obtain

$$\operatorname{dist}\big(E, \operatorname{spec}\big(H_{\Lambda_0}(x,\omega)\big)\big) < Ce^{-\ell^{1/4}} \ ,$$

as claimed (the same arguments applies to the larger intervals Λ_Q around n_0). From (6.9) of the previous lemma with $n = n_1 - Q$ (if $n_1 - Q < (\log N)^{2C_0}$, then proceed to the next case) one concludes from Cramer's rule, see (2.17), that

$$\big|G_{[1,n_1-Q]}(x,\omega,E)(k,m)\big| < \exp\big(-\gamma|k-m| + (\log N)^{C_0}\big) \tag{6.13}$$

for all $1 \le k, m \le n_1 - Q$. In particular,

$$\big|\psi_j^{(N)}(x,\omega;k)\big| < e^{-\frac{\gamma}{2}|n_1-Q-k|}$$

for all $1 \le k \le n_1 - 2Q$. The same reasoning applies to

$$G_{[n_1+Q,N]}(x,\omega,E)$$

via (6.10) of the previous lemma, and (ii) follows. For (i), note that (6.6) and (ii) imply that

$$\|(H_{\Lambda_Q}(x,\omega) - E_j^{(N)}(x,\omega))\psi_j^{(N)}\| \le e^{-\gamma Q/2}$$

Since $\|\psi_j^{(N)}\|_{\ell^2(\Lambda_Q)} \ge 1 - e^{\gamma Q}$, we obtain (i). □

Since eigenvalues of the Dirichlet problem are simple, we can order the $E_j^{(N)}(x,\omega)$ according to the convention

$$E_1^{(N)}(x,\omega) < E_2^{(N)}(x,\omega) < \ldots < E_N^{(N)}(x,\omega)$$

The following corollary deals with the stability of the localization statement of Lemma 6.4 with respect to the energy. As in previous stability results of this type in this paper, the most important issue is the relatively large size of the perturbation, i.e., $\exp(-(\log N)^C)$ instead of e^{-N}, say.

Corollary 6.5. *Using the assumptions and terminology of the previous proposition, let $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$, and $\nu_j^{(N)}(x,\omega)$ be associated with $\psi_j^{(N)}(x,\omega;\cdot)$ as stated there. If $|E - E_j^{(N)}(x,\omega)| < e^{-(\log N)^{40B}}$ with B as above, then*

$$\sum_{n=1}^{\nu_j^{(N)}(x,\omega)-Q} \big|f_{[1,n]}\big(e(x),\omega,E\big)\big|^2 < e^{-\gamma Q/2} \sum_{n\in\Lambda_Q} \big|f_{[1,n]}\big(e(x),\omega,E\big)\big|^2 \tag{6.14}$$

where $\Lambda_Q = \big[\nu_j^{(N)}(x,\omega) - Q, \nu_j^{(N)}(x,\omega) + Q\big] \cap [1,N]$. *Similarly,*

$$\sum_{n=\nu_j^{(N)}(x,\omega)+Q}^{N} \big|f_{[n,N]}(x,\omega,E)\big|^2 < e^{-\gamma Q/2} \sum_{n\in\Lambda_Q} \big|f_{[n,N]}(x,\omega,E)\big|^2 \tag{6.15}$$

Finally, under the same assumptions one has

$$\begin{aligned} &\big|f_{[1,n]}\big(e(x),\omega,E\big) - f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big| \\ &\le \exp\big((\log N)^C\big)\big|E - E_j^{(N)}\big(x,\omega)\big|\,\big|f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big| \end{aligned} \tag{6.16}$$

provided $1 \le n \le \nu_j^{(N)}(x,\omega) - Q$*, and similarly for* $f_{[n,N]}$.

Proof. For each j there exists a constant $\mu_j(x,\omega)$ so that

$$\psi_j^{(N)}(x,\omega;n) = \mu_j(x,\omega) f_{[1,n-1]}\big(x,\omega;E_j^{(N)}(x,\omega)\big)$$

for all $1 \le n \le N$ (with the convention that $f_{[1,0]} = 1$). A similar formula holds for

$$f_{[n+1,N]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big).$$

As in the previous proof, this implies estimate (6.13) with $E = E_j^{(N)}(x,\omega)$. Thus, for all $1 \le n \le \nu_j^{(N)}(x,\omega) - Q$ one has

$$\big|f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big| < e^{-\gamma|\nu_j^{(N)}(x,\omega)-n|/2}\ \big|f_{[1,\nu_j^{(N)}(x,\omega)]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|\ ,$$

which implies (6.14) for $E = E_j^{(N)}(x,\omega)$, and (6.15) follows by a similar argument for this E. Corollary 2.15 implies that

$$\begin{aligned} &\big|f_{[1,n]}\big(e(x),\omega,E\big) - f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big| \\ &\le \exp\big((\log N)^C\big)\big|E - E_j^{(N)}(x,\omega)\big|\ \big|f_{[1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big| \end{aligned}$$

for all $1 \le n \le \nu_j^{(N)}(x,\omega) - Q$, and (6.16) follows for all $\big|E - E_j^{(N)}(x,\omega)\big| < \exp\big(-(\log N)^B\big)$. ☐

7. Quantitative separation of the Dirichlet eigenvalues in finite volume

At least conceptually, this section provides arguably the most important single ingredient in the proof of gap formation. It already played a crucial role in our earlier work [GolSch2]. Based on the finite volume Anderson localization from the previous section, we shall now obtain a quantitative separation property of the eigenvalues on finite volume. Note carefully that *localization* does not depend on the off-diagonal terms in the Hamiltonian - indeed, a diagonal Hamiltonian has δ-function eigenstates which are perfectly localized. In contrast to this, the separation or even the simplicity of the eigenvalues of course crucially depend on these off-diagonal terms. Needless to say, the gaps in the spectrum also hinge on this property and this section is one of the places where it enters in an essential way. The reader will easily see this in the proofs of the first two results of this section. The off-diagonal terms enter there simply through the mechanism of transfer matrices; or in other words, we have to exploit that we are dealing with a second order difference equation which we can solve via initial conditions.

We now turn to the details. In this section it will be convenient for us to work with the operators $H_{[-N,N]}(x,\omega)$ instead of $H_{[1,N]}(x,\omega)$. Abusing our notation somewhat, we use the symbols $E_j^{(N)}, \psi_j^{(N)}$ to denote the eigenvalues and normalized eigenfunctions of $H_{[-N,N]}(x,\omega)$, rather than the eigenvalues and normalized eigenfunctions of $H_{[1,N]}(x,\omega)$, as in the previous section. A similar comment applies to $\Omega_N, \mathcal{E}_{N,\omega}$.

The following proposition states that the eigenvalues $\{E_j^{(N)}(x,\omega)\}_{j=1}^{2N+1}$ are separated from each other by at least e^{-N^δ} provided $\omega \notin \Omega_N$ and provided we delete those eigenvalues that fall into a bad set $\mathcal{E}_{N,\omega}$ of energies. We remind the reader that

$$\operatorname{mes}(\mathcal{E}_{N,\omega}) \lesssim \exp(-(\log N)^B), \quad \operatorname{compl}(\mathcal{E}_{N,\omega}) \lesssim N^3,$$

and similarly for Ω_N, see Lemma 6.2. This section corresponds to Section 11 of [GolSch2].

Proposition 7.1. *Fix $a > 1$, $c > 0$ and assume that $L(\omega, E) > \gamma > 0$ for all ω, E. Let Ω_N, $\mathcal{E}_{\omega,N}$ be as in Lemma 6.2. Furthermore, fix $\delta \in (0,1)$. Then there exists $N_0 = N_0(\delta, V, \rho_0, a, c, \gamma)$ so that for any $N \geq N_0$, any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ and all x one has*

$$\big|E_j^{(N)}(x,\omega) - E_k^{(N)}(x,\omega)\big| > e^{-N^\delta} \tag{7.1}$$

for all j,k provided $E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$.

Proof. Fix $x \in \mathbb{T}, E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}$. Let $Q \asymp \exp\big((\log\log N)^{B_1}\big)$, see Lemma 6.2. By Lemma 6.4 there exists

$$\Lambda_Q := \big[\nu_j^{(N)}(x,\omega) - Q, \nu_j^{(N)}(x,\omega) + Q\big] \cap [-N,N]$$

so that

$$\begin{aligned} &\sum_{n\in[-N,N]\setminus\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega;E_j^{(N)}(x,\omega)\big)\big|^2 \\ &< e^{-Q\gamma} \sum_{n=-N}^{N} \big|f_{[-N,n]}\big(e(x),\omega;E_j^{(N)}(x,\omega)\big)\big|^2 \ . \end{aligned} \tag{7.2}$$

Here we used that with some $\mu = \text{const}$

$$\psi_j^{(N)}(x,\omega;n) = \mu \cdot f_{[-N,n-1]}\big(e(x),\omega;E_j^{(N)}(x,\omega)\big)$$

for $-N \leq n \leq N$. Note the convention that

$$f_{[-N,-N-1]} = 0, \quad f_{[-N,-N]} = 1.$$

One can assume $\nu_j^{(N)}(x,\omega) \geq 0$ by symmetry. Using Corollary 2.15 and (6.16), we conclude that

$$\begin{aligned} &\sum_{n=-N}^{\nu_j^{(N)}(x,\omega)-Q} \big|f_{[-N,n]}(e(x),\omega,E) - f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \\ &\leq e^{-\gamma Q}\big|E - E_j^{(N)}(x,\omega)\big|^2 e^{(\log N)^C} \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \end{aligned} \tag{7.3}$$

Let $n_1 = \nu_j^{(N)}(x,\omega) - Q - 1$. Furthermore,

$$\begin{aligned} &\left\| \begin{pmatrix} f_{[-N,n+1]}(e(x),\omega,E) \\ f_{[-N,n]}(e(x),\omega,E) \end{pmatrix} - \begin{pmatrix} f_{[-N,n+1]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) \\ f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) \end{pmatrix} \right\| \\ &= \bigg\| M_{[n_1+1,n]}(e(x),\omega,E) \begin{pmatrix} f_{[-N,n_1+1]}(e(x),\omega,E) \\ f_{[-N,n_1]}(e(x),\omega,E) \end{pmatrix} \\ &\qquad - M_{[n_1+1,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) \begin{pmatrix} f_{[-N,n_1+1]}\big(e(x),\omega,E_j^{(N)}\big) \\ f_{[-N,n_1]}\big(e(x),\omega,E_j^{(N)}\big) \end{pmatrix} \bigg\| \\ &\leq e^{C(n-n_1)} e^{-\gamma Q/2}\big|E - E_j^{(N)}(x,\omega)\big| e^{(\log N)^C} \Big(\sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2\Big)^{\frac{1}{2}} \ . \end{aligned} \tag{7.4}$$

Now suppose there is $E_k^{(N)}(x,\omega)$ with $\big|E_k^{(N)}(e(x),\omega) - E_j^{(N)}(x,\omega)\big| < e^{-N^\delta}$ where the small $\delta > 0$ is arbitrary but fixed. Then (7.3), (7.4) imply that

$$\begin{aligned}(7.5)\qquad &\sum_{n=-N}^{\nu_j^{(N)}(x,\omega)+Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) - f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2 \\ &< e^{-\frac12 N^\delta} \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 ,\end{aligned}$$

provided $N^\delta > \exp\big((\log\log N)^{B_1}\big)$. Let us estimate the contributions of $\big[\nu_j^{(N)}(x,\omega)+Q, N\big]$ to the sum terms in the left-hand side of (7.5).

For both $E = E_j^{(N)}$ and $E_k^{(N)}$ one has

$$f_{[-N,n]}(e(x),\omega,E) = G_{[\nu_j^{(N)}(x,\omega)+\frac{Q}{2},N]}(e(x),\omega,E)\big(n,\nu_j^{(N)}(x,\omega)+\frac{Q}{2}\big) f_{[-N,\nu_j^{(N)}(x,\omega)+\frac{Q}{2}-1]}(e(x),\omega,E)$$

due to the zero boundary condition at $N+1$, i.e.,

$$f_{[-N,N]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) = f_{[-N,N]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big) = 0 .$$

Therefore,

$$(7.6)\qquad \sum_{n=\nu_j^{(N)}+Q}^{N} \big|f_{[-N,n]}(e(x),\omega,E)\big|^2 \le e^{-\frac{\gamma Q}{4}} \sum_{k\in\Lambda_Q} \big|f_{[-N,k]}(e(x),\omega,E)\big|^2$$

again for both $E = E_j^{(N)}(x,\omega)$ and $E = E_k^{(N)}(x,\omega)$. Finally, in view of (7.5) and (7.6),

$$\begin{aligned}(7.7)\qquad &\sum_{n=-N}^{N} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big) - f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2 \\ &< e^{-\frac{\gamma Q}{4}}\Big[\sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big|^2 \\ &\quad + \sum_{n\in\Lambda_Q} \big|f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big|^2\Big]\end{aligned}$$

By orthogonality of $\big\{f_{[-N,n]}\big(e(x),\omega,E_j^{(N)}(x,\omega)\big)\big\}_{n=-N}^N$ and $\big\{f_{[-N,n]}\big(e(x),\omega,E_k^{(N)}(x,\omega)\big)\big\}_{n=-N}^N$ we obtain a contradiction from (7.7). ☐

In order to capture the mechanism behind Figure 2 later in this paper, we will need to achieve the separation property of the previous proposition without removing any energies. Rather, we will be using some apriori information which has the same effect as requiring the energies to be "good". This is done in the following result which is a corollary of the preceding proof rather than of the statement itself.

Corollary 7.2. *Assume $L(\omega,E) > \gamma > 0$ for all ω, E and let $(x_0,\omega) \in \mathbb{T}\times\mathbb{T}_{c,a}$ be arbitrary where $a > 1$ and $c > 0$ are fixed. Moreover, fix constants $1 \ge \delta > \varepsilon > 0$ and let $N \ge N_0(\delta,\varepsilon,V,a,c,\gamma)$ be sufficiently large. Suppose there is $\Lambda = [N',N''] \subset [-N,N]$ satisfying*

$$(\log N)^{2C_0} \le |\Lambda| \le N^\varepsilon, \qquad 100(\log N)^{2C_0} < N' < N'' < N - 100(\log N)^{2C_0},$$

and such that for some pair $E_1, E_2 \in \mathbb{R}$

$$f_k(ze(s\omega),\omega,E) \ne 0 \quad \forall\, z\in\mathcal{D}(e(x_0),r_0),\ \forall\, E\in\mathcal{D}(E_1,r_0)\cup\mathcal{D}(E_2,r_0),\ \forall\, s\in[-N,N]\setminus\Lambda$$

and all choices of $(\log N)^{C_0} \le k \le 100(\log N)^{C_0}$ with $r_0 := \exp(-(\log N)^{C_0/2})$. If E_1, E_2 are eigenvalues of $H_{[-N,N]}(x_0,\omega)$, then

$$|E_1 - E_2| > e^{-N^\delta}$$

Furthermore, suppose $\psi_j(\cdot)$ are normalized eigenfunctions of the Dirichlet problem on $[-N,N]$

$$H_{[-N,N]}(x_0,\omega)\psi_j = E_j\,\psi_j$$

Then

$$\big|\psi_j(n)\big| \le \exp\big(-\gamma\,\mathrm{dist}(n,\Lambda)/2\big), \quad j=1,2 \tag{7.8}$$

for all $n\in[-N,N]$.

Proof. Let $\bar N' := N' + 10(\log N)^{C_0}$ and $\bar N'' := N'' - 10(\log N)^{C_0}$. We apply the avalanche principle Proposition 2.8 to $f_{[-N,\bar N']}(x_0,\omega,E)$ and $f_{[\bar N'',N]}(x_0,\omega,E)$ with arbitrary $E\in\mathcal{D}(E_1,r_0)\cup\mathcal{D}(E_2,r_0)$. It will suffice to consider the former. If for some choice of k as above and $-N\le s\le N'$

$$\log|f_k(x_0e(s\omega),\omega,E)| < kL(\omega,E) - k^{\frac34}$$

then by Corollary 2.18 one has $f_k(z_0e(s\omega),\omega,E)=0$ for some $z_0\in\mathcal{D}(e(x_0),e^{-\sqrt{k}})$ contradicting our hypothesis. Hence, Proposition 2.8 implies that

$$f_{[-N,\bar N']}(x_0,\omega,E)\neq 0 \quad \forall\, E\in\mathcal{D}(E_1,r_0)\cup\mathcal{D}(E_2,r_0)$$

and similarly for $f_{[-N,\bar N']}(x_0,\omega,E)$. In fact, by the same argument,

$$f_{[-N,\bar N']}(z,\omega,E)\neq 0 \quad \forall\, z\in\mathcal{D}(e(x_0),r_0),\ \forall\, E\in\mathcal{D}(E_1,r_0)\cup\mathcal{D}(E_2,r_0)$$

Now suppose that for some choice of $E\in\mathcal{D}(E_1,r_0)\cup\mathcal{D}(E_2,r_0)$,

$$\log|f_{[-N,\bar N']}(x_0,\omega,E)| < (\bar N'-N)L - (\log N)^{2C_0}$$

Then Corollary 2.18 implies that $f_{[-N,\bar N']}(z,\omega,E)=0$ where $|z-e(x_0)|<\exp(-(\log N)^{C_0})$, a contradiction to our choice of r_0. Thus,

$$\begin{aligned}\log|f_{[-N,\bar N']}(x_0,\omega,E)| &> (\bar N'-N)L - (\log N)^{2C_0}\\ \log|f_{[\bar N'',N]}(x_0,\omega,E)| &> (N-\bar N'')L - (\log N)^{2C_0}\end{aligned}$$

which in its turn imply the Green function bounds

$$\begin{aligned}|G_{[-N,\bar N']}(x_0,\omega,E)(p,q)| &\le \exp\big(-\gamma|p-q| + (\log N)^{2C_0}\big) \quad \forall\, p,q\in[-N,\bar N']\\ |G_{[\bar N'',N]}(x_0,\omega,E)(p,q)| &\le \exp\big(-\gamma|p-q| + (\log N)^{2C_0}\big) \quad \forall\, p,q\in[\bar N'',N]\end{aligned}$$

and some constant $C_0=C_0(a)$, see Cramer's rule (2.17). These bounds prove (7.8) via the Poisson formula (6.6). Furthermore, inspection of the proofs of Corollary 6.5 and Proposition 7.1 shows that they apply verbatim to the situation at hand (the only difference here is that e^{-N^δ} needs to beat e^{N^ε}). In particular, $|E_1-E_2|>e^{-N^\delta}$ for large N as desired. $\square$

The eigenvalues $E_j^{(N)}(x,\omega)$ of the Dirichlet problem on $[-N,N]$ are real-analytic functions of $x\in\mathbb{T}$ and can therefore be extended analytically to a complex neighborhood of $\mathbb{T}$. Moreover, by simplicity of the eigenvalues of the Dirichlet problem, the graphs of these functions of x do not cross. Proposition 7.1 makes this non-crossing quantitative, up to certain sections of the graphs where we lose control. These are the portions of the graph that intersect horizontal strips corresponding to energies in $\mathcal{E}_{N,\omega}$. The quantitative control provided by (10.1) allows us to give lower bounds on the radii of the disks to which the functions $E_j^{(N)}(x,\omega)$ extend analytically.

Corollary 7.3. *Fix $a>1$, $c>0$ as well as $\delta\in(0,1)$, assume $L(\omega,E)>\gamma>0$ for all (ω,E), and let Ω_N, $\mathcal{E}_{N,\omega}$ be as in Lemma 6.2. There exists a large integer $N_0(V,\gamma,a,c,\delta)$ such that for all $N\ge N_0$ the following holds: assume $f_N(x_0,\omega_0,E_0)=0$ for some $x_0\in\mathbb{T}$, $\omega_0\in\mathbb{T}_{c,a}\setminus\Omega_N$, and $E_0\in\mathbb{R}\setminus\mathcal{E}_{N,\omega_0}$. Then (with ω_0 fixed)*

$$f_N(z,\omega_0,E) = \big(E-b_0(z)\big)\chi(z,E) \tag{7.9}$$

for all $z\in\mathcal{D}(x_0,r_0)$, $E\in\mathcal{D}(E_0,r_1)$ where $r_1=e^{-N^\delta}$, $r_0=C^{-1}r_1$. Moreover, $b_0(z)$ is analytic on $\mathcal{D}(x_0,r_0)$, $\chi(z,E)$ is analytic and nonzero on $\mathcal{D}(x_0,r_0)\times\mathcal{D}(E_0,r_1)$, $b_0(x_0)=E_0$.

Proof. By Proposition 7.1, $f_N(x_0, \omega_0, E) \neq 0$ if $E \in \mathcal{D}(E_0, r_1)$, $E \neq E_0$. Since $H_N(x_0, \omega_0)$ is self adjoint and

$$\big\|H_N(z, \omega_0) - H_N(x_0, \omega_0)\big\| \lesssim |z - x_0|,$$

it follows that $f_N(z, \omega_0, E) \neq 0$ for any $|z - x_0| \leq C^{-1} r_1$, $\frac{r_1}{2} < |E - E_0| < \frac{3}{4} r_1$. The representation (7.9) is now obtained by the same arguments that lead to the Weierstrass preparation theorem, see Proposition 2.26. $\square$

We shall also require quantitative control on the function χ. Let $z_0 := e(x_0)$.

Corollary 7.4. *Using the notations of the previous corollary one has*

$$f_N(z, \omega_0, E) = (E - b_0(z))\chi(z, E)$$

where $\chi(z, E)$ is analytic in $\mathcal{D}(z_0, r_0) \times \mathcal{D}(E_0, r_0)$ and obeys the bound

$$NL(E_0, \omega_0) - N^{2\delta} \leq \log|\chi(z, E)| \leq NL(E_0, \omega_0) + N^{2\delta}$$

for any $(z, E) \in \mathcal{D}(z_0, r_0/2) \times \mathcal{D}(E_0, r_0/2)$.

Proof. Due to the uniform upper estimates on $\log|f_N|$ one has

$$\big|E - b_0(z)\big|\big|\chi(z, E)\big| \leq \exp(NL(E_0, \omega_0) + (\log N)^A)$$

for any $(z, E) \in \mathcal{D}(z_0, r_0) \times \mathcal{D}(E_0, r_0)$. Take arbitrary $z_1 \in \mathcal{D}(z_0, r_0)$, $E_1 \in \mathcal{D}(E_0, r_0/2)$. We distinguish two cases: (a) If

$$\big|E_1 - b_0(z_1)\big| \geq r_0/4$$

then

$$\big|\chi(z_1, E_1)\big| \leq 4\exp(NL(E_0, \omega_0) + N^{\delta}), \qquad \log|\chi(z_1, E_1)| \qquad \leq NL(E_0, \omega_0) + 2N^{\delta} \tag{7.10}$$

(b) Otherwise,

$$|E - b_0(z_1)| \geq r_0/4$$

for any $|E - E_1| = r_0/2$. Hence

$$|\chi(z_1, E)| \leq 4\exp(NL(E_0, \omega_0) + N^{\delta})$$

for any $|E - E_1| = r_0/2$ in this case. The maximum principle implies (7.10). Thus (7.10) holds for any $z_1 \in \mathcal{D}(z_0, r_0)$, $E_1 \in \mathcal{D}(E_0, r_0/2)$ which proves the upper estimate for $\log|\chi(z, E)|$. Furthermore, $|b_0(z)| \lesssim 1$ for any $z \in \mathcal{D}(z_0, r_0)$. Hence

$$\log|f_N(z, E)| \leq \log|\chi(z, E)| + C$$

It follows from the large deviation estimate that given $E_1 \in \mathcal{D}(E_0, r_0)$ there exists $z_1 \in \mathcal{D}(z_0, r_0/4)$ such that

$$\log|\chi(z_1, E_1)| > NL(E_0, \omega_0) - N^{2\delta}$$

Therefore, the lower bound for $\log|\chi(z, E)|$ follows from the Harnack inequalities. $\square$

8. Evaluating Jensen averages via the Harnack inequality

This section is part of the "zero counting machinery" from [GolSch2]. This machinery is of crucial importance to the gap development, as we shall see later. Technically speaking, the goal of this section is to develop estimates for $\log\|M_N\|$ that are analogous to those valid for $\log|f(z)|$ where f is analytic. Special attention will be paid to the location of the zeros of the entries of M_N. Similar considerations appear in [GolSch2], and as in that paper it will be convenient to work in the following degree of generality:

Definition 8.1. *Let $M(z)$ be a 2×2 matrix-function defined in a disk $\mathcal{D}(z_0, r_0) \subset \mathbb{C}$, $r_0 \ll 1$. Thus, let*

$$M(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} \tag{8.1}$$

$\det M(z) = 1$. *We say that $M(z)$ satisfies an* abstract large deviation estimate *(ALDE) provided the following holds: let $100 \le K := \sup\{\|M(z)\| : z \in \mathcal{D}(z_0, r_0)\} < \infty$. Then for any $H \ge (\log\log K)^C$ one has*

$$\log\big|a_{ij}(z)\big| > \log K - H \tag{ALDE}$$

for any entry a_{ij} which is not identically zero and all

$$z \in \mathcal{D}(z_0, r_0) \setminus \mathcal{B}, \quad \mathcal{B} = \bigcup_{j=1}^{J} \mathcal{D}(\zeta_j, r), \quad r = r_0 \exp\Big(\frac{-H}{(\log\log K)^C}\Big),$$

and $J \lesssim (\log\log K)^A$.

In our applications, A, C will be constants as in Definition 2.6. This definition is set up to be scaling invariant, which is useful throughout this section. We begin by recalling the following version of Harnack's inequality:

Lemma 8.2. *Let $f(x)$ be analytic in $\mathcal{D}(z_0, r_0)$ and non-vanishing in $\mathcal{D}(z_0, r_1)$ with $0 < r_1 \le r_0$. Assume that*

$$100 \le K := \sup\{|f(z)| : z \in \mathcal{D}(z_0, r_0)\} < \infty$$

Assume also that

$$|f(z_0)| \ge K^{-1} \tag{8.2}$$

Then, with some absolute constant C,

$$|f(\zeta)| \le C|f(z)|$$

for any $z, \zeta \in \mathcal{D}(z_0, r_2), r_2 = (1 + \log K)^{-2} r_1$.

Proof. The function $u(z) := \log K - \log|f(z)|$ is harmonic and non-negative in $\mathcal{D}(z_0, r_1)$. Applying Harnack's inequality to it in $\mathcal{D}(z_0, r_1)$ yields

$$\begin{aligned} &[1 - 2(1 + \log K)^{-2}](\log K - \log|f(z_0)|) \\ &\le \log K - \log|f(z)| \le [1 + 3(1 + \log K)^{-2}](\log K - \log|f(z_0)|) \end{aligned}$$

for any $z \in \mathcal{D}(z_0, r_2)$. Hence, using (8.2), this implies that

$$-2 - \log|f(z_0)| \le -\log|f(z)| \le 2 - \log|f(z_0)|$$

for any $z \in \mathcal{D}(z_0, r_2)$, and the lemma follows with $C = e^4$. □

Next, we turn to matrices as in Definition 8.1.

Lemma 8.3. *Fix some $0 < \delta_1 < \frac{1}{10}$. Then for K sufficiently large depending on δ_1 the following holds: suppose that one of the entries $a_{ij}(z)$ has no zeros in $\mathcal{D}(z_0, r_1)$, with $\exp(-(\log K)^{2\delta_1}) r_0 \le r_1 \le \exp(-(\log K)^{\delta_1}) r_0$. Then*

$$\Big|\log\frac{\|M(z)\|}{\|M(z_0)\|}\Big| \le \exp((\log K)^{5\delta_1})|z - z_0| r_0^{-1} \tag{8.3}$$

for any $|z - z_0| \le r_1$ for which the right-hand side of (8.3) is $< \frac{1}{2}$, say.

Proof. We first claim that

$$\log \|M(z_0)\| \geq \log K - (\log K)^{5\delta_1} \tag{8.4}$$

Let $a_{i_0 j_0}(z)$ be an entry which has no zeros in $\mathcal{D}(z_0, r_1)$. Due to condition (ALDE) there exists z_1 with $|z_1 - z_0| \leq r_0 \exp(-(\log K)^{3\delta_1})$ such that $|a_{i_0 j_0}(z_1)| \geq K \exp(-(\log K)^{4\delta_1}) > K^{-1}$. Since $a_{i_0 j_0}(z)$ is analytic and does not vanish in $\mathcal{D}(z_1, r_1/2)$, one can apply Lemma 8.2 with z_1 in the role of z_0. Therefore,

$$\log |a_{i_0 j_0}(z_1)| - C \leq \log |a_{i_0 j_0}(z_0)|$$

Hence,

$$\log |a_{i_0 j_0}(z_0)| \geq \log K - (\log K)^{5\delta_1}$$

which implies (8.4). Next, we note that for any $|z - z_0| < r_0$,

$$\begin{aligned} \|M(z) - M(z_0)\| &\leq |z - z_0| \sup_{|\zeta - z_0| \leq r_0} \|M'(\zeta)\| \leq 2|z - z_0| r_0^{-1} \sup_{|\zeta - z_0| \leq 2r_0} \|M(\zeta)\| \\ &\leq \exp((\log K)^{5\delta_1}) |z - z_0| r_0^{-1} \|M(z_0)\| \end{aligned}$$

where we used (8.4) to pass to the final inequality. This implies that

$$\Big| \frac{\|M(z)\|}{\|M(z_0)\|} - 1 \Big| \leq \exp\big((\log K)^{5\delta_1}\big) |z - z_0| r_0^{-1}$$

for all $|z - z_0| \ll r_1$, which is the same as (8.3). □

Next, we consider the case when all entries $a_{ij}(z)$ have zeros in $\mathcal{D}(z_0, r_0)$. Assume that for some $\zeta_0 \in \mathcal{D}(z_0, r_0/4)$ the following conditions[6] are valid[7]:

(a) each entry $a_{ij}(z)$ has exactly one zero in $\mathcal{D}(\zeta_0, \rho_0)$, where

$$\exp(-(\log K)^{2\delta_0}) r_0 \leq \rho_0 \leq \exp(-(\log K)^{\delta_0}) r_0$$

We denote this unique zero by ζ_{ij}.

(b) no entry $a_{ij}(z)$ has any zeros in $\mathcal{D}(\zeta_0, \rho_1) \setminus \mathcal{D}(\zeta_0, \rho_0)$, where

$$\exp(-(\log K)^{\delta_1}) r_0 \leq \rho_1 \leq r_0$$

with $0 < 10\delta_1 \leq \delta_0 \ll 1$.

K will need to be large depending on δ_0, δ_1.

Lemma 8.4. *The function $b_{ij}(z) := r_0 (z - \zeta_{ij})^{-1} a_{ij}(z)$ is analytic in $\mathcal{D}(z_0, r_0)$ and non-vanishing in $\mathcal{D}(\zeta_0, \rho_1)$. Set $\widetilde{M}(z) := \{b_{ij}(z)\}_{1 \leq i,j \leq 2}$. Then*

$$T := \sup\big\{ \|\widetilde{M}(z)\| : z \in \mathcal{D}(z_0, r_0/2) \big\} \leq 5K \tag{8.5}$$

Furthermore,

$$\log \|\widetilde{M}(z)\| \geq \log K - (\log K)^{3\delta_1} \tag{8.6}$$

for any $z \in \mathcal{D}(\zeta_0, \rho_2)$ with $\rho_2 = \rho_1 (\log K)^{-2}$, and

$$\log \|M(z)\| \geq \log K + \log(|z - \zeta_0| r_0^{-1}) - (\log K)^{4\delta_1} \tag{8.7}$$

for any $z \in \mathcal{D}(\zeta_0, \rho_2) \setminus \mathcal{D}(\zeta_0, 2\rho_0)$.

[6] We will later verify conditions (a) and (b) above for the Dirichlet determinants by means of Proposition 7.1.

[7] In this section ρ_0 is used with a different meaning than as before; however, there is no danger of confusion.

Proof. For every $|z - z_0| = r_0/2$ and large K,

$$|z - \zeta_{ij}| \geq |z - z_0| - |z_0 - \zeta_0| - |\zeta_0 - \zeta_{ij}| \geq r_0/4 - \rho_0 \geq r_0/5$$

which implies via the maximum principle that

$$|b_{ij}(z)| \leq 5|a_{ij}(z)| \leq 5K \quad \forall\, |z - z_0| \leq r_0/2$$

as claimed. Next, it follows from (ALDE) that for some $\zeta_1 \in \mathcal{D}(\zeta_0, \rho_1/2)$ one has

$$\log |b_{ij}(\zeta_1)| \geq \log K - (\log K)^{2\delta_1}$$

Applying Lemma 8.2 above with ζ_1 in the role of z_0 and $\rho_1/2$ in the role of r_1 implies that

$$|b_{ij}(z)| \geq \log K - (\log K)^{3\delta_1}$$

for all $z \in \mathcal{D}(\zeta_0, \rho_2)$ where ρ_2 is as above. Finally, (8.7) follows from (8.6) since

$$\begin{aligned}\log(|z - \zeta_{ij}| r_0^{-1}) &= \log(|z - \zeta_0| r_0^{-1}) - \log(|z - \zeta_0||z - \zeta_{ij}|^{-1}) \\ &\geq \log(|z - \zeta_0| r_0^{-1}) - \log(1 + \rho_0 |z - \zeta_{ij}|^{-1}) \geq \log(|z - \zeta_0| r_0^{-1}) - \log 2\end{aligned}$$

since $|z - \zeta_{ij}| \geq |z - \zeta_0| - |\zeta_0 - \zeta_{ij}| \geq \rho_0$ for all $z \in \mathcal{D}(\zeta_0, \rho_2) \setminus \mathcal{D}(\zeta_0, 2\rho_0)$. □

Next, we obtain the analogue of Lemma 8.3 for the case of $M(z)$ as in (a), (b) above.

Lemma 8.5. *For any $z, w \in \mathcal{D}(\zeta_0, \rho_4) \setminus \mathcal{D}(\zeta_0, \rho_3)$ one has*

$$\Big| \log \frac{\|M(z)\|}{\|M(w)\|} - \log \frac{|z - \zeta_0|}{|w - \zeta_0|} \Big| \leq C \exp(-(\log K)^{\delta_2}) \tag{8.8}$$

where $\rho_4 = \exp(-(\log K)^{2\delta_2})\rho_1$ and $\rho_3 = \exp((\log K)^{\delta_2})\rho_0$, where $1 > \delta_0 \geq \delta_2 \geq 3\delta_1 > 0$ and K is large depending on these parameters.

Proof. Set

$$\hat{a}_{ij}(z) := (z - \zeta_0) r_0^{-1}\, b_{ij}(z),\ i, j = 1, 2, \quad \widehat{M}(z) := \big\{\hat{a}_{ij}(z)\big\}_{1 \leq i,j \leq 2}$$

Then

$$\log \frac{\|M(z)\|}{\|M(w)\|} - \log \frac{|z - \zeta_0|}{|w - \zeta_0|} = \log \Big\{ \frac{\|M(z)\|}{\|\widehat{M}(z)\|} \frac{\|\widehat{M}(w)\|}{\|M(w)\|} \Big\} + \log \frac{\|\widetilde{M}(z)\|}{\|\widetilde{M}(w)\|}$$

Since

$$\frac{\|M(z)\|}{\|\widehat{M}(z)\|} \leq 1 + \frac{\|M(z) - \widehat{M}(z)\|}{\|\widehat{M}(z)\|} \leq 1 + \max_{1 \leq i,j \leq 2} \frac{|\zeta_0 - \zeta_{ij}|}{|z - \zeta_0|} \leq 1 + \frac{\rho_0}{\rho_3}$$

and similarly,

$$\frac{\|M(z)\|}{\|\widehat{M}(z)\|} \geq 1 - \frac{\|M(z) - \widehat{M}(z)\|}{\|\widehat{M}(z)\|} \geq 1 - \max_{1 \leq i,j \leq 2} \frac{|\zeta_0 - \zeta_{ij}|}{|z - \zeta_0|} \geq 1 - \frac{\rho_0}{\rho_3}$$

we see that

$$\Big| \log \Big\{ \frac{\|M(z)\|}{\|\widehat{M}(z)\|} \frac{\|\widehat{M}(w)\|}{\|M(w)\|} \Big\} \Big| \leq C \frac{\rho_0}{\rho_3}$$

whereas from Lemma 8.3 one has

$$\Big| \log \frac{\|\widetilde{M}(z)\|}{\|\widehat{M}(w)\|} \Big| \leq C \exp\big((\log K)^{3\delta_1}\big) |z - w| r_0^{-1}$$

for any $z, w \in \mathcal{D}(\zeta_0, \rho_2)$ where ρ_2 is as above. In conclusion,

$$\Big| \log \frac{\|M(z)\|}{\|M(w)\|} - \log \frac{|z - \zeta_0|}{|w - \zeta_0|} \Big| \leq C \exp\big((\log K)^{3\delta_1}\big) \rho_4 r_0^{-1} + C \frac{\rho_0}{\rho_3}$$

which implies the lemma. □

Next, we apply these results to the propagator matrices M_N. Of particular importance for the zero count are the *Jensen averages*

$$\mathcal{J}(u,z,r) = \fint_{\mathcal{D}(z,r)} [u(\zeta) - u(z)]\,d\xi d\eta \tag{8.9}$$

$$\mathcal{J}(u,\mathcal{D},r_2) = \fint_{\mathcal{D}} \mathcal{J}(u,z,r_2)\,dx\,dy \tag{8.10}$$

where $\mathcal{D}$ is an arbitrary bounded domain in the second line (as usual, $\fint$ means average). Recall that we are assuming that V is analytic on some annulus $\mathcal{A}_{\rho_0}$.

Proposition 8.6. *Let $0 < 10\delta_1 < \delta_0$ be fixed small parameters, $\omega \in \mathbb{T}_{c,a}$, and $z_0 \in \mathcal{A}_{\rho_0/2}$. There exists a positive integer $N_0(\delta_0, \delta_1, c, a, \gamma, V, E)$ so that the following holds:*

(1) *Suppose that one of the Dirichlet determinants*

$$f_{[1,N]}(\cdot,\omega,E),\ f_{[1,N-1]}(\cdot,\omega,E),\ f_{[2,N]}(\cdot,\omega,E),\ f_{[2,N-1]}(\cdot,\omega,E)$$

has no zeros in $\mathcal{D}(z_0,r_1)$, $\exp(-N^{2\delta_1}) \le r_1 \le \exp(-N^{\delta_1})$. Then

$$\Big|\log\frac{\|M_N(z,\omega,E)\|}{\|M_N(z_0,\omega,E)\|}\Big| \le C\exp(N^{5\delta_1})|z-z_0| \tag{8.11}$$

for any $z \in \mathcal{D}(z_0, r_1/2)$. In particular, with $\mathcal{J}(u,z,r)$ defined as in (8.9) one has

$$\mathcal{J}\big(\log\|M_N(\cdot,\omega,E)\|, z_0, r\big) \le r\exp(N^{5\delta_1}) \tag{8.12}$$

for any $e^{-\sqrt{N}} \le r \le r_1/2$.

(2) *Assume that for some $\zeta_0 \in \mathcal{A}_{\rho_0/8}$ the following conditions are valid:*

(a) *each determinant*

$$f_{[1,N]}(\cdot,\omega,E),\ f_{[1,N-1]}(\cdot,\omega,E),\ f_{[2,N]}(\cdot,\omega,E),\ f_{[2,N-1]}(\cdot,\omega,E)$$

has exactly one zero in $\mathcal{D}(\zeta_0,\rho_0)$, where $\exp(-N^{2\delta_0}) \le \rho_0 \le \exp(-N^{\delta_0})$

(b) *none of these determinants has any zeros in $\mathcal{D}(\zeta_0,\rho_1)\setminus\mathcal{D}(\zeta_0,\rho_0)$, where $\exp(-N^{\delta_1}) \le \rho_1 \le \rho_0/10$.*

Then

$$\Big|\log\frac{\|M_N(z,\omega,E)\|}{\|M_N(\zeta,\omega,E)\|} - \log\frac{|z-\zeta_0|}{|\zeta-\zeta_0|}\Big| \le \exp(-N^{\delta_2}) \tag{8.13}$$

for any

$$\rho_0\exp(N^{\delta_2}) \le |z-\zeta_0|, |w-\zeta_0| \le \rho_1\exp(-N^{2\delta_2})$$

where $\frac{1}{4} > \delta_0 \ge \delta_2 \ge 3\delta_1 > 0$. Furthermore,

$$\Big|\mathcal{J}\big([\log\|M_N(\cdot,\omega,E)\| - \log|\cdot-\zeta_0|], z, r\big)\Big| \le \exp(-N^{2\delta_2}) \tag{8.14}$$

for any $z \in \mathcal{D}(\zeta_0,\rho_1/2)$ and any $\rho_0\exp(N^{\delta_2}) \le r \le \rho_1\exp(-N^{2\delta_2})$. Statements similar to (1), (2) hold with respect to zeros in the E-variable with $z = e(x)$ arbitrary but fixed.

Proof. Part (1) follows from Lemma 8.3 with a choice of $r_0 = \rho_0/2$. Part (2) follows from Lemma 8.5. In both cases, the (ALDE) holds due to the large deviation theorem for M_N and f_N, see Propositions 2.7 (for the LDE in x) and 2.23 and Remark 2.24 (for the LDE in E). $\square$

9. A multi-scale approach to counting zeros of Dirichlet determinants

The basic question motivating this section is as follows: suppose

$$[1,N) = \bigcup_{j=1}^{J-1} [n_0, n_{j+1}), \quad 1 = n_0 < n_1 < \ldots < n_J = N$$

Can we relate the number of zeros of $f_{[1,N)}(\cdot,\omega,E)$ in $\mathcal{D}(z_0,r)$ to the sum of the numbers of zeros of $f_{[n_j,n_{j+1})}(\cdot,\omega,E)$ in $\mathcal{D}(z_0,r)$? This seems like a very farfetched question; indeed, since typically

$$f_{[1,N)}(\cdot,\omega,E) \neq \prod_{j=0}^{J-1} f_{[n_j,n_{j+1})}(\cdot,\omega,E) \tag{9.1}$$

there is no reason to assume that the zeros on the left-hand side are in any way related to the zeros on the right-hand side. Nevertheless, we shall see in this section that under certain conditions (which will still be flexible enough for our purposes) such an addition theorem does hold for the number of zeros. The basic tools here are the avalanche principle and the Jensen averages from Section 2. The former will give us something akin to (9.1), whereas the latter allows for an effective zero count based on averaging. Averaging here is particularly important as it "washes out" a set of exceptional phases that need to be removed for the avalanche principle to hold. Results similar to those of this section can be found in Sections 12 and 13 of [GolSch2]. We describe how to combine Proposition 8.6 with the avalanche principle expansion to count precisely the number of the zeros of Dirichlet determinants. The following definition is very important in this regard. For the following definition recall that $\mathcal{Z}(f,z,r) = \{\zeta \in \mathcal{D}(z,r) : f(\zeta) = 0\}$.

Definition 9.1. *Let $\ell \geq 1$ be some integer, and $s \in \mathbb{Z}$. Fix (ω,E) as well as some disk $\mathcal{D}(z_0,r_0)$. We say that s is* adjusted *to $(\mathcal{D}(z_0,r_0),\omega,E)$ at scale ℓ if for all $\ell \leq k \leq 100\ell$*

$$\mathcal{Z}(f_k(\cdot e((s+m)\omega),\omega,E),z_0,r_0) = \emptyset \qquad \forall\, |m| \leq 100\ell.$$

First, an easy but useful observation: the determinants appearing in the definition of "adjusted" automatically satisfy a large deviation type estimate.

Lemma 9.2. *Let $\omega_0 \in \mathbb{T}_{c,a}$ and $E_0 \in \mathbb{C}$. There exists $\ell_0 = \ell_0(V,\rho_0,a,c)$ so that if s is adjusted to $(\mathcal{D}(z_0,r_0),\omega_0,E_0)$ at scale $\ell \geq \ell_0$ with $r_0 > e^{-\sqrt{\ell}}$, then*

$$\log|f_k(ze((s+m)\omega_0),\omega_0,E_0)| > kL(\omega_0,E_0) - k^{\frac{3}{4}} \quad \forall\, |z-z_0| < r_0/2$$

for all $|m| \leq 100\ell$ and $\ell \leq k \leq 100\ell$.

Proof. Suppose not. Then

$$\log|f_k(z_1e((s+m)\omega_0),\omega_0,E_0)| < kL(\omega_0,E_0) - k^{\frac{3}{4}}$$

for some choice of $|m| \leq 100\ell$, $\ell \leq k \leq 100\ell$, and $|z_1 - z_0| < r_0/2$. By Corollary 2.18, there exists

$$|z_2 - z_1| < \exp\big(-k^{\frac{3}{4}}/(\log k)^{C_0}\big) < r_0/2$$

such that $f_k(z_2e((s+m)\omega_0),\omega_0,E_0) = 0$. But this contradicts Definition 9.1. □

We shall now prove that the notion of "adjusted" allows for an affirmative answer to the "additivity of the zero count" question stated at the beginning of this section. In the following proposition, the constants implicit in the $\ll$ and $\lesssim$ notations are absolute. For the ν_f notation, see (4.2). Also, as usual V is analytic on $\mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$.

Proposition 9.3. *Let $a > 1, c > 0$ and fix $\omega_0 \in \mathbb{T}_{c,a}$. Assume that $L(\omega_0,E_0) > \gamma > 0$ where $E_0 \in \mathbb{C}$ is arbitrary but fixed. There exists a large integer $N_0 = N_0(V,\rho_0,\gamma,a,c,E_0)$ such that for any $N \geq N_0$*

the following holds. Let ℓ be an integer $(\log N)^A \le \ell$ where $A = A(V, \rho_0, \gamma, a, c, E_0)$ is a large constant. Suppose that with some $n_0 := 1 < n_1 < n_2 < \ldots < n_J < n_{J+1} := N$,

$$[1, N] = \bigcup_{j=1}^{J} [n_{j-1}, n_j) \cup [n_J, N], \quad m := \min_{0\le j\le J} (n_{j+1} - n_j) > 10\ell$$

We assume that n_j is adjusted to $(\mathcal{D}(z_0, r_1), \omega_0, E_0)$ at scale ℓ for each $0 \le j \le J+1$, where $z_0 = e(x_0)$, $x_0 \in \mathbb{T}$, and $e^{-\sqrt{\ell}} < r_1 < \rho_0$. Let $\Lambda_j := [n_j, n_{j+1})$ with $0 \le j \le J-1$ and $\Lambda_J := [n_J, N]$. Then, with $e^{-\sqrt{m}} < r_0 < N^{-1} r_1$ and $r_2 = C^{-1} r_0$,

$$\mathcal{J}(\log |f_{[1,N]}(\cdot, \omega_0, E_0)|, z_0, r_0, r_2) = \sum_{j=1}^{J} \mathcal{J}(\log |f_{\Lambda_j}(\cdot, \omega_0, E_0)|, z_0, r_0, r_2) + O(N\ell^{-1}\big(r_0 r_1^{-1} + e^{-\ell^{\frac{1}{2}}}\big))$$

Furthermore, suppose also that for all $0 \le j \le J$ one has

$$\mathcal{Z}\big(f_{\Lambda_j}(\cdot, \omega_0, E_0), \mathcal{D}(z_0, 3r_0/2) \setminus \mathcal{D}(z_0, r_0/2)\big) = \emptyset \tag{9.2}$$

Then

$$\nu_{f_{[1,N]}(\cdot,\omega_0,E_0)}(z_0, r_0) = \sum_{j=0}^{J} \nu_{f_{\Lambda_j}(\cdot,\omega_0,E_0)}(z_0, r_0)$$

Finally, if every $1 \le s \le N$ is adjusted to $(\mathcal{D}(z_0, r_1), \omega_0, E_0)$ at scale ℓ, then $\nu_{f_N(\cdot,\omega_0,E_0)}(z_0, N^{-1} r_1) = 0$.

Proof. We begin by noting that

$$\begin{bmatrix} f_N(z, \omega_0, E_0) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_N(z, \omega_0, E_0) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The idea is to apply the avalanche principle to the matrix on the right-hand side by writing it as a product of monodromy matrices corresponding to the Λ_j. However, we need to connect any two such adjacent matrices by (much shorter) ones of length ℓ. Hence, we let

$$\Lambda_j' := [n_j + 2\ell, n_{j+1} - 2\ell], \quad A_j(z) := M_{\Lambda_j'}(z, \omega_0, E_0), \quad 0 \le j \le J$$

where M_Λ for an interval $\Lambda \subset \mathbb{Z}$ denotes the monodromy matrix corresponding to Λ. Next, we define

$$B_{j,1}(z) := M_{(n_j-2\ell, n_j-\ell]}(z, \omega_0, E_0), \quad B_{j,2}(z) := M_{(n_j-\ell, n_j]}(z, \omega_0, E_0),$$
$$B_{j,3}(z) := M_{(n_j, n_j+\ell)}(z, \omega_0, E_0), \quad B_{j,4}(z) := M_{[n_j+\ell, n_j+2\ell)}(z, \omega_0, E_0)$$

for $1 \le j \le J$ and

$$B_{0,3}(z) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{[1,\ell)}(z, \omega_0, E_0), \quad B_{J+1,2}(z) := M_{(N-\ell,N]}(z, \omega_0, E_0) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

as well as $B_{0,4}(z) := M_{[\ell,2\ell)}(z, \omega_0, E_0)$, $B_{J+1,1}(z) := M_{(N-2\ell,N-\ell]}(z, \omega_0, E_0)$. Then,

$$\begin{bmatrix} f_N(z, \omega_0, E_0) & 0 \\ 0 & 0 \end{bmatrix} = B_{0,3}(z) B_{0,4}(z) A_0(z) \prod_{j=1}^{J} \Big(B_{j,1}(z) B_{j,2}(z) B_{j,3}(z) B_{j,4}(z) A_j(z)\Big) B_{J+1,1}(z) B_{J+1,2}(z)$$

Since each n_j is adjusted to $(\mathcal{D}(z_0, r_1), \omega_0, E_0)$ at scale ℓ, Lemma 9.2 implies that each $B_{j,i}$ satisfies an estimate of the form

$$\log \|B_{j,i}(z)\| > \ell L(\omega_0, E_0) - \ell^{\frac{3}{4}} \quad \forall\, z \in \mathcal{D}(z_0, r_1/2)$$

The point here is that we avoid the removal of sets of measure $e^{-\sqrt{\ell}}$ in the z-variable coming from the large deviation theorem; such sets would be unnecessarily large. On the other hand, the large deviation theorem applied to each A_j implies that there exists $\mathcal{B} := \mathcal{B}_{N,\omega_0,E_0} \subset \mathbb{C}$ with $\mathrm{mes}\,(\mathcal{B}) \le \exp\big(-m^{3/4}\big)$ so that for any $z \in \mathcal{D}(z_0, r_1/2) \setminus \mathcal{B}$

$$\log \|A_j(z)\| > |\Lambda_j| L(\omega_0, E_0) - |\Lambda_j|^{\frac{7}{8}}$$

Therefore, for all $z \in \mathcal{D}(z_0, r_1/2) \setminus \mathcal{B}$, one has the avalanche principle expansion

$$
\begin{aligned}
\log\big|f_N(z,\omega_0,E_0)\big| &= \log\|B_{0,1}(z)B_{0,2}(z)\| + \log\|B_{0,2}(z)A_0(z)\| + \log\|A_0(z)B_{1,1}(z)\| + \dots \\
&\quad + \log\|B_{n,4}A_n(z)\| + \log\|A_nB_{n+1,1}(z)\| + \log\|B_{n+1,1}(z)B_{n+1,2}(z)\| \\
&\quad - \Big(\log\|B_{0,2}(z)\| + \log\|A_0(z)\| + \dots + \log\|B_{1,1}(z)\| + \dots + \log\|B_{n,4}(z)\| + \\
&\quad + \log\|A_n(z)\| + \log\|B_{n+1,1}(z)\| + \log\|B_{n+1,2}(z)\|\Big) + O(e^{-\ell^{1/2}})
\end{aligned} \tag{9.3}
$$

Next, we apply the avalanche principle again, this time to each of the $f_{\Lambda_j'}(z,\omega_0,E_0)$, $0 \le j \le J$. Thus, we write

$$
\begin{bmatrix} f_{\Lambda_j}(z,\omega_0,E_0) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{\Lambda_j}(z,\omega_0,E_0) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{B}_{j,3}(z)B_{j,4}(z)A_j(z)B_{j+1,1}(z)\widetilde{B}_{j+1,2}(z)
$$

where

$$
\widetilde{B}_{j,3}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M_{[n_j, n_j+\ell)}(z,\omega_0,E_0), \qquad \widetilde{B}_{j,2}(z) := M_{[n_j-\ell, n_j)}(z,\omega_0,E_0) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

Thus, for all $z \in \mathcal{D}(z_0, r_1/2) \setminus \mathcal{B}$,

$$
\begin{aligned}
&\log|f_{\Lambda_j}(z,\omega_0,E_0)| = \log\|\widetilde{B}_{j,3}(z)B_{j,4}(z)\| + \log\|B_{j,4}(z)A_j(z)\| + \log\|A_j(z)B_{j+1,1}(z)\| \\
&+ \log\|B_{j+1,1}(z)\widetilde{B}_{j+1,2}(z)\| - \Big(\log\|B_{j,4}(z)\| + \log\|A_j(z)\| + \log\|B_{j+1,1}(z)\|\Big) + O(e^{-\sqrt{\ell}})
\end{aligned}
$$

where we again applied Lemma 9.2. Summing these expressions over j, and subtracting the result from (9.3) shows the following: for all $z \in \mathcal{D}(z_0, r_1/2) \setminus \mathcal{B}$

$$
\log\big|f_N(z,\omega_0,E_0)\big| - \sum_{j=0}^{n} \log\big|f_{\Lambda_j}(z,\omega_0,E_0)\big| = \sum_{k=1}^{K} \pm \log\|W_k(z)\| + O(e^{-\sqrt{\ell}}) \tag{9.4}
$$

with $K \le N$ and W_k being a 2×2-matrix each entry of which is either identically zero or a determinant $f_\Lambda(z,\omega_0,E_0)$ with $\Lambda \subset [1,N]$ being an interval of length proportional to ℓ; the point here is that we set up the avalanche principle in such a way that the bulk terms containing $A_j(z)$ exactly cancel in (9.4). Moreover, every W_k contains at least one nonzero entry, which necessarily is a determinant satisfying the conditions of Definition 9.1, i.e., it does not vanish on $\mathcal{D}(z_0, r_1)$. Thus, by Proposition 8.6, and with $r_2 := r_0/10$,

$$
\Big|\mathcal{J}\big(\log|f_N(\cdot,\omega_0,E_0)|, z_0, r_0, r_2\big) - \sum_{j=0}^{J} \mathcal{J}(\log|f_{\Lambda_j}(\cdot,\omega_0,E_0)|, z_0, r_0, r_2)\Big| \lesssim N\ell^{-1}\big(r_0 r_1^{-1} + e^{-\ell^{\frac12}}\big) \tag{9.5}
$$

We used here that $J \le N\ell^{-1}$. It is important to note that r_2 is very large compared to the measure of $\mathcal{B}$. Hence, when applying the averaging operator $\mathcal{J}$ the exceptional set $\mathcal{B}$ only produces an additive error of the form $e^{-\ell^{\frac12}}$. By (9.5) and Corollary 4.2,

$$
\nu_{f_N(\cdot,\omega_0,E_0)}(z_0, r_0 - r_2) \le \nu_g(z_0, r_0 + r_2), \qquad \nu_g(z_0, r_0 - r_2) \le \nu_{f_N(\cdot,\omega_0,E_0)}(z_0, r_0 + r_2),
$$

where

$$
g(z) := \prod_{j=0}^{J} f_{\Lambda_j}\big(z,\omega_0,E_0\big)
$$

Replacing r_0 by $(r_0 \pm r_2)$, one obtains similarly

$$
\nu_{f_N(\cdot,\omega_0,E_0)}(z_0, r_0) \le \nu_g(z_0, r_0 + 2r_2), \qquad \nu_g(z_0, r_0 - 2r_2) \le \nu_{f_N(\cdot,\omega_0,E_0)}(z_0, r_0)
$$

Due to the assumptions of the lemma

$$
\nu_g(z_0, r_0 + 2r_2) = \nu_g(z_0, r_0 - 2r_2)
$$

and the assertion follows. ☐

Heuristically speaking, the fact that edges of the main intervals Λ_j are adjusted in Proposition 9.3 ensures that these intervals are "independent" of each other. This is of course crucial for the zero count to work. One can think of it in this way: each zero z_{jk} of $\det(H_{\Lambda_j}(\cdot,\omega_0)-E)$ produces an eigenstate ψ_{jk} on Λ_j with energy E_0 that is localized strictly inside Λ_j (since by adjustedness the Green function decays exponentially close to the edges of Λ_j). For this reason, we can simply extend each of these eigenstates $\psi_{jk}(z)$ to all of $[1,N]$ by setting them equal to zero outside of Λ_j. This creates an approximate eigenstate of $H_{[1,N]}(x_0,\omega_0)$ with energy E_0 and therefore a zero z' of $\det(H_{[1,N]}(\cdot,\omega_0)-E)$ which is very close to z_{jk}. Note that in this process the assumption (9.2) plays a very important role: it guarantees that we do not "pull in" any zero from outside of $\mathcal{D}(z_0,r_0)$ — this may certainly occur under the process we just described. This discussion shows that the assumptions of Proposition 9.3 are essentially optimal. We now consider the exact same questions but with regard to zeros in E rather than z. Just as in the case of zeros in z there is a notion of "adjusted" for zeros in E.

Definition 9.4. *Let $\ell \geq 1$ be some integer, and $s \in \mathbb{Z}$. Fix (z,ω). We say that s is* adjusted *to $(z,\omega,\mathcal{D}(E_0,r_0))$ at scale ℓ if for all $\ell \leq k \leq 100\ell$*

$$\mathcal{Z}(f_k(ze((s+m)\omega),\omega,\cdot),E_0,r_0)=\emptyset \qquad \forall\, |m|\leq 100\ell.$$

Using self-adjointness of $H_\Lambda(x,\omega)$, for $x\in\mathbb{R}$ and real-valued V we can say that the notion of "adjusted" is symmetric with respect to z and E. This will use the crucial Corollary 2.18.

Lemma 9.5. *Let V be real-valued on $\mathbb{T}$ and let $z_0 = e(x_0)$ where $x_0\in\mathbb{R}$. There exists a constant $C(V)$ with the following property: suppose that s is adjusted to $(z_0,\omega_0,\mathcal{D}(E_0,r_0))$ at scale ℓ. Then s is adjusted to $(z,\omega_0,\mathcal{D}(E_0,r_0))$ at scale ℓ for every $z\in\mathcal{D}(z_0,C(V)^{-1}r_0)$. In other words, s is adjusted to $(\mathcal{D}(z_0,C(V)^{-1}r_0),\omega_0,E)$ at scale ℓ for every $E\in\mathcal{D}(E_0,r_0)$. In particular,*

$$\log|f_k(e(z+(s+m)\omega_0),\omega_0,E)| > kL(\omega_0,E)-k^{\frac{3}{4}} \quad \forall\,|E-E_0|<r_0/2,\ \forall\,|z-z_0|<C(V)^{-1}r_0$$

and for all $|m|\leq 100\ell$, $\ell\leq k\leq 100\ell$. Conversely, if s is adjusted to $(\mathcal{D}(z_0,r_0),\omega_0,E_0)$ at scale ℓ, then s is adjusted to $(z_0,\omega_0,\mathcal{D}(E_0,r_1))$ at scale ℓ, where $\log(r_1^{-1})=\log(r_0^{-1})(\log\ell)^{C_2}$.

Proof. We need to show that for all $\ell\leq k\leq 100\ell$

$$\mathcal{Z}(f_k(ze((s+m)\omega_0),\omega_0,\cdot),E_0,r_0/2)=\emptyset \qquad \forall\,|m|\leq 100\ell$$

and all $|z-z_0|\leq C(V)^{-1}r_0$. Suppose $f_k(ze((s+m)\omega_0),\omega_0,E)=0$ for some choice of $|z-z_0|\leq C(V)^{-1}r_0$, $E\in\mathcal{D}(E_0,r_0/2)$, and k,m in the admissible ranges. Since for large enough $C(V)$

$$\big\|H_{[1,k]}(ze((s+m)\omega_0),\omega_0)-H_{[1,k]}(z_0e((s+m)\omega_0),\omega_0)\big\|\leq C_1(V)|z-z_0|<r_0/2$$

and $H_{[1,k]}(z_0e((s+m)\omega_0),\omega_0)$ is Hermitian, we conclude by Lemma 2.19 that the latter operator would need to have an eigenvalue in the interval $(E-r_0/2,E+r_0/2)$. Since this is included in $\mathcal{D}(E_0,r_0)$, we arrive at a contradiction to Definition 9.4. The final statement follows from Lemma 9.2.

For the converse, note that if s is adjusted to $(\mathcal{D}(z_0,r_0),\omega_0,E_0)$ at scale ℓ, then for any $\ell\leq k\leq 100\ell$

$$\log|f_k(z_0,\omega_0,E_0)| > kL(E_0,\omega_0)-\log(r_0^{-1})(\log k)^{C_0}$$

by Corollary 2.18. Next, by Corollary 2.15, $f_k(z_0,\omega_0,E)\neq 0$ for all $|E-E_0|<r_1$ where r_1 is as in the statement of the lemma. ☐

We are now in a position to state the analogue of Proposition 9.3 with regard to the E-variable. Recall that the proof of that result used the large deviation estimate. Here, we shall do the same but with regard to the matrix function $E\mapsto M_N(e(x),\omega,E)$. The large deviation estimate for this purpose is provided by Proposition 2.23.

Proposition 9.6. *Let V be real-valued and $a>1, c>0$ and fix $\omega_0\in\mathbb{T}_{c,a}$. Assume that $L(\omega_0,E_0)>\gamma>0$ where $E_0\in\mathbb{C}$ is arbitrary but fixed. There exists a large integer $N_0=N_0(V,\rho_0,\gamma,a,c,E_0)$ such that for*

any $N \ge N_0$ the following holds. Let ℓ be an integer such that $(\log N)^A \le \ell$ where $A = A(V, \rho_0, \gamma, a, c, E_0)$ is a large constant. Suppose that with some $n_0 := 1 < n_1 < n_2 < \ldots < n_J < n_{J+1} := N$,

$$[1, N] = \bigcup_{j=1}^{J} [n_{j-1}, n_j) \cup [n_J, N], \quad m := \min_{0 \le j \le J} (n_{j+1} - n_j) > 10\ell$$

Suppose moreover that n_j is adjusted to $(z_0, \omega_0, \mathcal{D}(E_0, r_1))$ at scale ℓ for each $0 \le j \le J+1$, where $z_0 = e(x_0)$, $x_0 \in \mathbb{T}$, and $e^{-\ell^{\frac14}} < r_1 < \exp(-(\log \ell)^{C_0})$. Let $\Lambda_j := [n_j, n_{j+1})$ with $0 \le j \le J-1$ and $\Lambda_J := [n_J, N]$. Let $e^{-m^{\frac14}} < r_0 < N^{-1} r_1$ be arbitrary. Then, with $r_2 = C^{-1} r_0$,

$$\mathcal{J}(\log |f_{[1,N]}(z_0, \omega_0, \cdot)|, E_0, r_0, r_2) = \sum_{j=1}^{J} \mathcal{J}(\log |f_{\Lambda_j}(z_0, \omega_0, \cdot)|, E_0, r_0, r_2) + O(N \ell^{-1} \big(r_0 r_1^{-1} + e^{-\ell^{\frac12}}\big))$$

Furthermore, suppose that for all $0 \le j \le J$ one has

$$\mathcal{Z}\big(f_{\Lambda_j}(z_0, \omega_0, \cdot), \mathcal{D}(E_0, 3r_0/2) \setminus \mathcal{D}(E_0, r_0/2)\big) = \emptyset$$

Then

$$\nu_{f_{[1,N]}(z, \omega_0, \cdot)}(E_0, r_0) = \sum_{j=0}^{J} \nu_{f_{\Lambda_j}(z, \omega_0, \cdot)}(E_0, r_0) \qquad \forall\, |z - z_0| < C(V)^{-1} r_0$$

Finally, if every $1 \le s \le N$ is adjusted to $(z_0, \omega_0, \mathcal{D}(E_0, r_1))$ at scale ℓ, then $\nu_{f_N(\cdot, \omega_0, E_0)}(z, N^{-1} r_1) = 0$ for all $|z - z_0| < C(V)^{-1} N^{-1} r_1$.

Proof. This is very similar to the proof of Proposition 9.3. More precisely, running the argument of Proposition 9.3 in the variable E rather than z yields the following:

$$\nu_{f_{[1,N]}(z, \omega_0, \cdot)}(E_0, u r_0) = \sum_{j=0}^{J} \nu_{f_{\Lambda_j}(z, \omega_0, \cdot)}(E_0, u r_0)$$

for all $x \in \mathbb{T}$ where $\frac45 < u < \frac65$. Therefore, using Lemma 2.19 yields that

$$\nu_{f_{[1,N]}(z, \omega_0, \cdot)}(E_0, r_0) = \sum_{j=0}^{J} \nu_{f_{\Lambda_j}(z, \omega_0, \cdot)}(E_0, r_0)$$

for all $|z - z_0| < C(V)^{-1} r_0$ as claimed. □

In applications we will need to chose the n_j to be adjusted. This is can be done via the following results.

Lemma 9.7. *Let $\omega_0 \in \mathbb{T}_{c,a}$, $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$, and $n_0 \in \mathbb{Z}$. Given $\ell \gg 1$ and $r_1 = \exp(-(\log \ell)^C)$, there exists*

$$n_0' \in [n_0 - \ell^6, n_0 + \ell^6] \tag{9.6}$$

such that with $z_0 = e(x_0)$,

$$f_k\big(\cdot e(n\omega_0), \omega_0, E_0\big) \text{ has no zero in } \mathcal{D}(z_0, r_1) \tag{9.7}$$

for any $|n - n_0'| \le 100\ell$ and $\ell \le k \le 100\ell$. In other words, each n_0' is adjusted to $(z_0, \omega_0, \mathcal{D}(E_0, r_2))$ with $r_2 = \exp(-(\log \ell)^{2C})$.

Proof. Suppose this fails. Then there exists a sequence $\{k_j\}_{j=1}^J \subset [\ell, 100\ell]$ with $J \ge \ell^4$ as well as an increasing sequence $\{n_j\}_{j=1}^J \subset [n_0 - \ell^6, n_0 + \ell^6]$ so that

$$\mathcal{Z}(f_{k_j}(\cdot e(n_j \omega_0), \omega_0, E_0)) \cap \mathcal{D}(z_0, r_1) \neq \emptyset$$

for each $1 \le j \le J$. Since there are at most 100ℓ choices for k_j, there exists some j_0 in this range such that

$$\mathcal{Z}(f_{k_{j_0}}(\cdot e(m_i \omega_0), \omega_0, E_0)) \cap \mathcal{D}(z_0, r_1) \neq \emptyset$$

for some increasing sequence $\{m_i\}_{i=1}^{J'} \subset [n_0 - \ell^6, n_0 + \ell^6]$ where $J' \ge \ell^2$. Since $f_{k_{j_0}}(\cdot, \omega_0, E_0)$ has at most $C(V)\ell$ zeros, it follows that there exists $z_1 \in \mathcal{D}(z_0, r_1)$ as well as $m \in [1, 2\ell^6]$ with the property that $z_1 e(m\omega_0) \in \mathcal{D}(z_0, r_1)$. However, this contradicts the Diophantine property of ω_0. □

This lemma gives us a lot of room to find adjusted sequences.

Corollary 9.8. *Assume that $\omega_0 \in \mathbb{T}_{c,a}$. Given a disk $\mathcal{D}(z_0, r_1)$, $r_1 \asymp \exp(-(\log \ell)^A)$ and an increasing sequence $\{\tilde{n}_j\}_{j=1}^{j_0}$ such that $\tilde{n}_{j+1} - \tilde{n}_j > \ell^7$ for $1 \le j < j_0$, there exists an increasing sequence $\{n_j\}_{j=1}^{j_0}$ which is adjusted to $\mathcal{D}(z_0, r_1)$ at scale ℓ and such that*

$$\left| n_j - \tilde{n}_j \right| < \ell^6, \quad 1 \le j \le j_0. \tag{9.8}$$

Proof. Simply apply the previous lemma to each $\tilde{n}_j$. □

We now show how one can apply this zero count to improving the bound on the separation between the zeros as in Proposition 5.5. The point is that due to passing to a smaller scale we will be able to substantially reduce the size of t in Proposition 5.5.

Proposition 9.9. *Assume that $L(\omega, E) \ge \gamma > 0$ for all[8] ω, E. Given $c > 0$, $a > 1$, and $A > 1$ there exists $N_0 = N_0(V, c, a, \gamma, A)$ such that for any $N \ge N_0$ and $T \ge 2N$ there exist $\Omega_{N,T} \subset \mathbb{T}$ and $\mathcal{E}_{N,\omega,T} \subset \mathbb{R}$ with*

$$\operatorname{mes}(\Omega_{N,T}) \le T \exp(-(\log N)^A), \quad \operatorname{compl}(\Omega_{N,T}) \le T^2 N$$

$$\operatorname{mes}(\mathcal{E}_{N,\omega,T}) \le T \exp(-(\log N)^A), \quad \operatorname{compl}(\mathcal{E}_{N,\omega,T}) < T^2 N$$

and with the following property: for any $\omega \in \mathbb{T}_{c,a} \setminus \Omega_{N,T}$, $z_0 = e(x_0)$, $E_0 \in \mathbb{R} \setminus \mathcal{E}_{N,\omega,T}$ and any N', t which satisfy the following conditions

(i) $(\log \min(N, N'))^{C_0} \ge \log \max(N, N')$
(ii) $2N \le t \le T$,

one has

$$\mathcal{Z}(f_N(z_0, \omega, \cdot), \mathcal{D}(E_0, r_0)) \cap \mathcal{Z}(f_{N'}(z_0 e(t\omega), \omega, \cdot), \mathcal{D}(E_0, r_0)) = \emptyset$$

where $r_0 := \exp(-(\log N)^A)$.

Proof. Let ℓ be an integer, $\ell = (\log N)^{4A}$. Let $\Omega_{\ell_1,\ell_2,t',H}$, and $\mathcal{E}_{\ell_1,\ell_2,t',H,\omega}$ be as in Proposition 5.5. Set $H = \ell^{1/4}$,

$$\Omega_{N,T} = \bigcup_{N \le t' \le T} \bigcup_{\ell \le \ell_1, \ell_2 \le 100\ell} \Omega_{\ell_1,\ell_2,t',H}$$

$$\mathcal{E}_{N,\omega,T} = \bigcup_{N \le t' \le T} \bigcup_{\ell \le \ell_1, \ell_2 \le 100\ell} \mathcal{E}_{\ell_1,\ell_2,t',H,\omega}$$

Assume that $\mathcal{Z}(f_N(z_0, \omega, \cdot), \mathcal{D}(E_0, r_0)) \ne \emptyset$. Due to the last part of Proposition 9.6 there exists $1 \le s \le N$ which is not adjusted to $(z_0, \omega_0, \mathcal{D}(E_0, r_1))$ at scale ℓ, where $r_1 := \exp(-\ell^{7/24})$; in other words,

$$\mathcal{Z}(f_{\ell_1}(z_0 e(s\omega), \omega, \cdot), \mathcal{D}(E_0, r_1)) \ne \emptyset$$

for some $\ell \le \ell_1 \le 100\ell$. Then, due to Proposition 5.5, provided $\omega \in \mathbb{T}_{c,a} \setminus \tilde{\Omega}_{N,T}$ and $E_0 \in \mathbb{R} \setminus \tilde{\mathcal{E}}_{N,\omega,T}$ one has

$$\mathcal{Z}(f_{\ell_2}(z_0 e(s'\omega), \omega, \cdot), \mathcal{D}(E_0, r_1)) = \emptyset$$

for any $2N \le s' \le N'$. We used here that $t' := s' - s > N$, and $N > \exp((\log\log \ell)^{C_0})$, where C_0 is the same as int he statement of Proposition 5.5. That suffices for Proposition 5.5. Hence

$$\mathcal{Z}(f_{N'}(z_0 e(t\omega), \omega, \cdot), \mathcal{D}(E_0, r_0)) = \emptyset$$

due to last part of Proposition 9.6. □

[8] One can localize here as usual.

Next, we use this improvement in the size of t to reduce the size of the window of localization in Section 6. The gain here is due to a "induction on scales" which enters into the proof of the previous proposition through the zero count used there (Proposition 9.9). We shall use the notations of that section as for example $\nu_j^{(N)}(x,\omega)$.

Corollary 9.10. *Assume that $L(\omega,E)\ge\gamma>0$ for all ω,E. Given $c>0$, $a>1$, A there exists $N_0=N_0(V,\gamma,a,c,A)$ such that for any $N\ge N_0$ there exist $\Omega_N\subset\mathbb{T}$, $\Omega'_N\subset\mathbb{T}$, $\mathcal{E}_{N,\omega}\subset\mathbb{R}$, $\mathcal{E}'_{N,\omega}\subset\mathbb{R}$ with*

$$\begin{aligned}
\operatorname{mes}(\Omega_N)&\le\exp(-(\log N)^A),\ \ \operatorname{compl}(\Omega_N)\le N^4\\
\operatorname{mes}(\Omega'_N)&\le\exp(-(\log\log N)^A),\ \ \operatorname{compl}(\Omega'_N)\le\exp((\log\log N)^{A/2})\\
\operatorname{mes}(\mathcal{E}_{N,\omega})&\le\exp(-(\log N)^A),\ \ \operatorname{compl}(\mathcal{E}_{N,\omega})<N^4\\
\operatorname{mes}(\mathcal{E}'_{N,\omega})&\le\exp(-(\log\log N)^A),\ \ \operatorname{compl}(\mathcal{E}'_{N,\omega})<\exp((\log\log N)^{A/2})
\end{aligned}\tag{9.9}$$

satisfying the following properties: for any $\omega\in\mathbb{T}_{c,a}\setminus(\Omega_N\cup\Omega'_N)$ and any $x\in\mathbb{T}$, any ℓ^2-normalized eigenfunction $\psi_j^{(N)}(x,\omega)$ of $H_{[-N,N]}(x,\omega)$ with associated eigenvalue $E_j^{(N)}(x,\omega)\in\mathbb{R}\setminus(\mathcal{E}_{N,\omega}\cup\mathcal{E}'_{N,\omega})$ satisfies

$$|\psi_j^{(N)}(x,\omega)(n)|\le C\exp(-\gamma\operatorname{dist}(n,\Lambda_j)/2)$$

for all $n\in[-N,N]$ where $\Lambda_j:=[\nu_j^{(N)}(x,\omega)-\ell,\nu_j^{(N)}(x,\omega)+\ell]\cap[-N,N]$ where $\ell=(\log N)^{4A}$.

Proof. Inspection of the proofs in Section 6 shows that the size of the window of localization is determined by the size of the shift t that assures separation of the zeros as in Proposition 9.9. Note that we apply that proposition on scale ℓ rather than N; the point here is that we then take $T=\exp((\log\log N)^{B_1})$ which is the size of the localization window guaranteed by Proposition 6.4 from Section 6. As long as we choose $A\gg B_1$ the corollary immediately follows. $\square$

Definition 9.11. *Using the notations of the previous corollary, we set*

$$\Omega_N^{(1)}:=\Omega_N\cup\Omega'_N,\quad \mathcal{E}_{N,\omega}^{(1)}:=\mathcal{E}_{N,\omega}\cup\mathcal{E}'_{N,\omega}$$

In what follows we shall use this notation for sets satisfying the estimates from (9.9)*. We shall also need to go down one more level: thus, set*

$$\Omega_N^{(2)}:=\Omega_N^{(1)}\cup\Omega_\ell^{(1)},\quad \mathcal{E}_{N,\omega}^{(2)}:=\mathcal{E}_{N,\omega}^{(1)}\cup\mathcal{E}_{\ell,\omega}^{(1)}$$

where $\ell=(\log N)^{4A}$ as in Corollary 9.10.

10. On the parametrization of the Dirichlet eigenfunctions

In this section we describe the graphs of the Rellich parametrization of the eigenvalues and eigenfunctions. It should be thought of as a preliminary ingredient in the construction of Sinai's function Λ, see Section 13. In this section we shall assume for simplicity that

$$L(\omega,E)>\gamma>0\quad\forall(\omega,E)\in\mathbb{T}\times\mathbb{R}$$

This assumption can of course be localized to a rectangle $(\omega',\omega'')\times(E',E'')$. In addition, we will fix $a>1$, $c>0$ and consider $\mathbb{T}_{c,a}$.

Proposition 10.1. *Given $0<\delta<1$ there exist large constants $N_0=N_0(\delta,V,\gamma,a,c)$, and $A=A(\delta,V,\gamma,a,c)$ such that for any $N\ge N_0$, and any $(\log N)^A=\ell$ there exist $\Omega_N^{(1)}$, $\mathcal{E}_{N,\omega}^{(1)}$ as in Definition 9.11 such that for any $\omega\in\mathbb{T}_{c,a}\setminus\Omega_N^{(1)}$ and all $x\in\mathbb{T}$ one has*

$$\big|E_j^{(N)}(x,\omega)-E_k^{(N)}(x,\omega)\big|>\exp(-\ell^\delta)\tag{10.1}$$

for all $j\ne k$ provided $E_j^{(N)}(x,\omega)\notin\mathcal{E}_{N,\omega}^{(1)}$.

Proof. This follows from the proof of the eigenvalue separation from Section 7 in combination with the reduction of the size of the localization window which was obtained in Corollary 9.10. $\square$

We emphasize that $\delta A \gg 1$. Thus, the separation achieved here is always much smaller than N^{-1}.

Corollary 10.2. *Using the notations of Definition 9.11, assume that $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(2)}$ and $E_{j_k}(x,\omega) \notin \tilde{\mathcal{E}}_{N,\omega}^{(2)}$, $k = 1,2$ for some $x \in \mathbb{T}$. If for $j_1 \neq j_2$*

$$|\nu_{j_1}^{(N)}(x,\omega) - \nu_{j_2}^{(N)}(x,\omega)| \le \ell$$

then

$$|E_{j_1}^{(N)}(x,\omega) - E_{j_2}^{(N)}(x,\omega)| \ge \exp(-(\log \ell)^A)$$

where A is a large parameter as in the definition.

Proof. Since $|\nu_{j_1}^{(N)}(x,\omega) - \nu_{j_2}^{(N)}(x,\omega)| \le \ell$,

$$|\psi_{j_s}^{(N)}(x,\omega,n)| \le \exp(-\gamma|n - \nu_{j_1}^{(N)}(x,\omega)|/2)$$

for $|n - \nu_{j_1}^{(N)}(x,\omega)| \ge C\ell$, $s = 1,2$, due to Corollary 9.10 where $C \gg 1$. Hence,

$$\operatorname{dist}\left[E_{j_s}^{(N)}(x_0,\omega), \operatorname{spec}\left(H_{[\nu_{j_1}^{(N)}(x,\omega)-C\ell,\nu_{j_1}^{(N)}(x,\omega)+C\ell]}(x,\omega)\right)\right] \le \exp(-\gamma\ell) \tag{10.2}$$

and the corollary follows from the previous Proposition 10.1. Indeed, that proposition, applied to

$$H_{[\nu_{j_1}^{(N)}(x,\omega)-C\ell,\nu_{j_1}^{(N)}(x,\omega)+C\ell]}(x,\omega)$$

guarantees a splitting between the eigenvalues $E_{j_s}^{(N)}$ by an amount $\exp(-(\log \ell)^{A\delta})$. Since $A\delta \gg 1$ anyway, we simply set $\delta = 1$ and we are done. Note that (10.2) guarantees that the restriction to the interval $[\nu_{j_1}^{(N)}(x,\omega) - C\ell, \nu_{j_1}^{(N)}(x,\omega) + C\ell]$ only affects the estimate by an exponentially small amount $e^{-\gamma\ell}$ which is acceptable. $\square$

We will now investigate – and single out – those portions of the graphs of the eigenvalues $E_j^{(N)}(x,\omega)$ which have controlled slopes and controlled separations from the other eigenvalues. In order to obtain reasonable complexity bounds (i.e., to efficiently limit the number of these portions), we replace the function $V(e(x))$, $x \in \mathbb{T}$ by an approximating algebraic polynomial. Since $V(z)$ is analytic in $\mathcal{A}_{\rho_0}$ the Fourier coefficients $\hat{v}(n)$ of $v(x) := V(e(x))$ satisfy

$$|\hat{v}(n)| \le B_0 \exp\big(-\frac{\rho_0}{2}|n|\big)$$

where $B_0 = \max\{|V(z)| : z \in \mathcal{A}_{\rho_0/2}\}$. Replacing the exponent $e(nx)$ by their Taylor polynomials leads to the following statement.

Lemma 10.3. *Given $0 < \sigma < 1$, and $T \ge 1$ there exists a polynomial $\widetilde{V}(x), x \in \mathbb{R}$ with real coefficients such that*

(1) $\max\limits_{|x| \le T} |V(e(x)) - \widetilde{V}(x)| \le \sigma$
(2) $\deg \widetilde{V}(x) \le 4T(K_0 + \log \sigma^{-1})$, $K_0 = K_0(V)$

For the remainder of this section, we set $\sigma := \sigma_N = \exp(-N^2)$, and $T := T_N = N+1$ and we denote the corresponding polynomial from Lemma 10.3 by $\widetilde{V}_N(x)$. Let $\tilde{H}_{[-N,N]}(x,\omega)$ be the Schrödinger operator on $[-N,N]$ with Dirichlet boundary conditions and with potential $\widetilde{V}_N(x+n\omega)$, $-N \le n \le N$. Furthermore, let

$$\widetilde{E}_1^{(N)}(x,\omega) < \widetilde{E}_2^{(N)}(x,\omega) < \cdots < \widetilde{E}_{2N+1}^{(N)}(x,\omega)$$

be the eigenvalues of $\tilde{H}_{[-N,N]}(x,\omega)$.

Lemma 10.4. *With the previous notation, one has the following estimates:*

(1) $\|H_{[-N,N]}(x,\omega) - \widetilde{H}_{[-N,N]}(x,\omega)\| \le \sigma, \quad \forall\, x,\omega \in [-1,1]$

(2) *For any $x, \omega \in [-1,1]$, one has $1 \le j \le 2N+1$ there exists $1 \le j_1 \le 2N+1$ such that*
$$|E_j^{(N)}(x,\omega) - \widetilde{E}_{j_1}^{(N)}(x,\omega)| \le \sigma,$$
and vice versa.

(3) *If for some x, ω*
$$\min_{1 \le j < 2N} \big(E_{j+1}^{(N)}(x,\omega) - E_j^{(N)}(x,\omega)\big) \ge \exp(-N^\delta),$$
then one has $j_1 = j$ and
$$|E_j^{(N)}(x,\omega) - \widetilde{E}_j^{(N)}(x,\omega)| \le \sigma, \ \ j \in [1, 2N+1]$$
$$\min_{1 \le j \le 2N} (\widetilde{E}_{j+1}^{(N)}(x,\omega) - \widetilde{E}_j^{(N)}(x,\omega)) \ge \frac{1}{2}\exp(-N^\delta)$$

Proof. (1) follows from the estimate (1) of Lemma 10.3. Properties (2) and (3) now follow from assertion (1) due to basic facts about perturbations of self-adjoint operators. □

Set
$$\tilde{f}_N(e(x), \omega, E) := \det(\tilde{H}_{[-N,N]}(x,\omega) - E)$$
Note that due to property (1) of Lemma 10.3, $\tilde{f}_N(x,\omega,E)$ is a polynomial in x, ω, E with
$$\deg \tilde{f}_N \le N^4, \quad \text{for} \quad N \ge N_0(V,c,a,\gamma) \tag{10.3}$$
Given an arbitrary interval $[\underline{E}, \overline{E}]$ set
$$\begin{aligned} \mathcal{B}_{N,\omega}(\underline{E}, \overline{E}) &= \{x \in [0,1] : \operatorname{spec}\big(H_{[-N,N]}(x,\omega)\big) \cap (\underline{E}, \overline{E}) \ne \emptyset\} \\ \widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E}) &= \{x \in [0,1] : \operatorname{spec}\big(\widetilde{H}_{[-N,N]}(x,\omega)\big) \cap (\underline{E}, \overline{E}) \ne \emptyset\} \end{aligned} \tag{10.4}$$
These sets have a number of simple properties:

Lemma 10.5. *The sets introduced in* (10.4) *satisfy the following properties:*

(1) $\widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E}) \subset \mathcal{B}_{N,\omega}(\underline{E} - \sigma, \overline{E} + \sigma) \subset \widetilde{\mathcal{B}}_{N,\omega}(\underline{E} - 2\sigma, \overline{E} + 2\sigma)$

(2) $\operatorname{mes} \widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E}) \le \exp(-H/(\log N)^C)$, *where $e^{-H} = \min(1/2, \overline{E} - \underline{E})$*

(3) $\widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E})$ *consists of a union of at most $O(K_0 N^4)$ closed intervals, where K_0 is as in Lemma 10.3*

Proof. Property (1) is due to fact (2) of Lemma 10.4. (2) follows from (1) due to Lemma 2.17 (the analogue of Wegner's estimate). The edges of the maximal intervals in the complement of $\widetilde{\mathcal{B}}_{N,\omega}$ are the roots of the equations
$$\tilde{f}_N(x,\omega,\underline{E}) = 0 \quad \text{or} \quad \tilde{f}_N(x,\omega,\overline{E}) = 0$$
Since $\deg \tilde{f}_N \lesssim K_0 N^4$, property (3) follows. □

For the rest of this section, we fix $\omega \in \mathbb{T}_{c,a} \backslash \Omega_N$, where $\Omega_N, \mathcal{E}_{N,\omega}$ are the sets from Corollary 7.2 and Proposition 7.1..

Corollary 10.6. *There exist intervals $[\xi'_k, \xi''_k] \subset [0,1]$, $k = 1, \dots, k_0$ such that the following conditions hold:*

(1) $|E_j^{(N)}(x,\omega) - \widetilde{E}_j^{(N)}(x,\omega)| \le \sigma$ *for any $x \in \bigcup_k [\xi'_k, \xi''_k]$ and any $j = 1, \dots, 2N+1$*

(2)
$$\min_{1 \le j \le 2N} \big(E_{j+1}^{(N)}(x,\omega) - E_j^{(N)}(x,\omega)\big) \ge \frac{1}{4}\exp(-N^\delta)$$
$$\min_{1 \le j \le 2N} \big(\widetilde{E}_{j+1}^{(N)}(x,\omega) - \widetilde{E}_j^{(N)}(x,\omega)\big) \ge \frac{1}{4}\exp(-N^\delta)$$
for any $x \in \bigcup_k [\xi'_k, \xi''_k]$

(3) $\operatorname{mes}\big([0,1] \backslash \bigcup_k [\xi'_k, \xi''_k]\big) \le \exp(-\tfrac{1}{2}(\log N)^B)$

(4) $k_0 \lesssim N^7$

Proof. Set

$$\mathcal{E}^{(+)}_{N,\omega} := \left\{E \in \mathbb{R} \,:\, \mathrm{dist}(E, \mathcal{E}_{N,\omega}) < \sigma_N\right\}$$
$$\widetilde{\mathcal{B}}_{N,\omega} := \left\{x \in [0,1] \,:\, \mathrm{spec}\left(\tilde{H}_{[-N,N]}(x,\omega)\right) \bigcap \mathcal{E}^{(+)}_{N,\omega} \neq \emptyset\right\}$$

Note that

$$\widetilde{\mathcal{B}}_{N,\omega} = \bigcup_{[\underline{E},\overline{E}] \subset \mathcal{E}^{(+)}_{N,\omega}} \widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E})$$

where the union here runs over the maximal subintervals of $\mathcal{E}^{(+)}_{N,\omega}$. One the one hand, due to the properties of $\mathcal{E}_{N,\omega}$, the set $\mathcal{E}^{(+)}_{N,\omega}$ can be covered by at most N^3 intervals $[E'_k, E''_k]$ with

$$\sum_k (E''_k - E'_k) \lesssim \exp(-(\log N)^B)$$

On the other hand, by Lemma 10.5, it follows that for each such k, the set $\widetilde{\mathcal{B}}_{N,\omega}(\underline{E}, \overline{E})$ is the union of at most $\lesssim N^4$ intervals on the x-axis. Hence, $\widetilde{\mathcal{B}}_{N,\omega}$ is the union of at most $\lesssim N^7$ intervals. The maximal intervals in the complement of $\widetilde{\mathcal{B}}_{N,\omega}$ are now defined to be $[\xi'_k, \xi''_k]$ with $1 \le k \le k_0$. The corollary follows by combining Lemma 10.4 and Lemma 10.5. $\square$

We also record the following standard fact about the perturbations of analytic matrix functions $M(z)$ which take values in the Hermitian matrices. We state it for the case of $H_{[-N,N]}(x,\omega)$ with N large.

Lemma 10.7. *Assume that for some x_0, ω_0, j_0*

$$\min_{j \neq j_0} |E^{(N)}_j(x_0,\omega_0) - E^{(N)}_{j_0}(x_0,\omega_0)| \ge \sigma^{(0)} > 0 \tag{10.5}$$

Then there exists an analytic function $E^{(N)}_{j_0}(z,\omega), (z,\omega) \in \mathcal{D}(x_0,r_0) \times \mathcal{D}(\omega_0,r_0), r_0 = \sigma^{(0)}_2/N^2$ such that

$$\mathrm{spec}\left(H_{[-N,N]}(z,\omega)\right) \cap \mathcal{D}(E^{(N)}_{j_0}, \sigma^{(0)}/2) = \{E^{(N)}_{j_0}(z,\omega)\}$$

for any $(z,\omega) \in \mathcal{D}(x_0,r_0) \times \mathcal{D}(\omega_0,r_0)$. Furthermore, suppose (10.5) *holds for all j_0. If $\frac{1}{2}\sigma^{(0)} > \sigma_N$, then for each j there exists an analytic function $\widetilde{E}^{(N)}_j(z,\omega), (z,\omega) \in \mathcal{D}(x_0,r_0) \times \mathcal{D}(\omega_0,r_0)$ such that*

$$\mathrm{spec}(\widetilde{H}_{[-N,N]}(z,\omega)) \cap \mathcal{D}(\widetilde{E}^{(N)}_j(x_0,\omega_0), \sigma^{(0)}/2) = \{\widetilde{E}^{(N)}_j(z,\omega)\}$$

for any $(z,\omega) \in \mathcal{D}(x_0,r_0) \times \mathcal{D}(\omega_0,r_0)$. Finally, for each j,

$$|E^{(N)}_j(z,\omega) - \widetilde{E}^{(N)}_j(z,\omega)| \le 2\sigma_N$$
$$|\partial^\alpha E^{(N)}_j(z,\omega) - \partial^\alpha \widetilde{E}_j(z,\omega)| \le 2\alpha!(\frac{r_0}{2})^{-|\alpha|}\sigma_N$$

for any $(z,\omega) \in \mathcal{D}(x_0,r_0/2) \times \mathcal{D}(\omega_0,r_0/2)$.

Combining Lemma 10.7 with Corollary 10.6 one obtains the following.

Corollary 10.8. *Using the notations of Corollary 10.6 one has*

$$|\partial_x E^{(N)}_j(x,\omega) - \partial_x \widetilde{E}^{(N)}_j(x,\omega)| \le \sqrt{\sigma_N}$$

for any $x \in \bigcup_k [\xi'_k, \xi''_k]$ and any $j = 1, \ldots, 2N+1$.

Note that each $\widetilde{E}^{(N)}_j(\cdot,\omega)$ is an algebraic function (see Appendix A). That implies the following statement.

Lemma 10.9. *For each $j = 1, \ldots, 2N+1$ and each $\tau > 0$ there exist disjoint intervals $[\eta'_{j,m}(\tau), \eta''_{j,m}(\tau)]$, $m = 1, 2, \ldots, m_0$, $m_0 \le N^9$ such that*

(i) $|\partial_x \widetilde{E}_j^{(N)}(x,\omega)| > \tau$ *for any* $x \in [-1,1] \setminus \bigcup_m [\eta'_{j,m}(\tau), \eta''_{j,m}(\tau)]$

(ii) $|\partial_x \widetilde{E}_j^{(N)}(x,\omega)| \le \tau$ *for any* $x \in \bigcup_m [\eta'_{j,m}(\tau), \eta''_{j,m}(\tau)]$

Proof. The degree of $\tilde f_N$ is $\lesssim N^4$, see (10.3). Since $\tilde f_N(x,\omega,\widetilde{E}_j^{(N)}(x,\omega)) = 0$, we see that

$$\partial_x \widetilde{E}_j^{(N)}(x,\omega) = -\partial_E \tilde f_N(x,\omega,E)/\partial_x \tilde f_N(x,\omega,E)\Big|_{E=\widetilde{E}_j^{(N)}(x,\omega)}$$

Hence, by Bezout's theorem in the appendix it follows that the equation $\partial_x \widetilde{E}_j^{(N)}(x,\omega) = \pm\tau$ has at most $2N^8$ solutions, whence the result. $\square$

Lemma 10.10. *Using the notations of Lemma 10.9 define*

$$\underline{E}_j(\tau,m) := \min\{\widetilde{E}_j^{(N)}(x,\omega) : x \in [\eta'_{j,m}(\tau), \eta''_{j,m}(\tau)]\}$$
$$\overline{E}_j(\tau,m) := \max\{\widetilde{E}_j^{(N)}(x,\omega) : x \in [\eta'_{j,m}(\tau), \eta''_{j,m}(\tau)]\}$$

$m = 1,\dots,m_0$. *Then one has*

$$\overline{E}_j(\tau,m) - \underline{E}_j(\tau,m) \le 2\tau \tag{10.6}$$

$$\eta''_{j,m}(\tau) - \eta'_{j,m}(\tau) \le \exp(-\log\tau^{-1}/(\log N)^C) \tag{10.7}$$

Proof. The estimate (10.6) follows from part (ii) of Lemma 10.9. The bound (10.7) follows from Relation (10.6) due to Lemma 2.17 (which is the analogue of Wegner's estimate). $\square$

Now we obtain the main result of this section. It allows one to control the graphs of the eigenvalues in terms of slopes and separation properties up to the removal of certain sets.

Proposition 10.11. *Assume that* $L(\omega,E) \ge \gamma > 0$ *for all* $\omega \in (\omega',\omega'')$ *and all* $E \in (E',E'')$. *Given* $\delta \ll 1 \ll A$ *there exists* $N_0 = N_0(V,c,a,\gamma,\delta,A)$, *such that for any* $N \ge N_0$, *and arbitrary* $\exp(-N^\delta) < \tau < \exp(-(\log N)^A)$ *there exist* $\mathcal{B}_{N,\omega}$, $\mathcal{B}'_{N,\omega}$ $\mathcal{E}_{N,\omega}(\tau), \mathcal{B}_{N,\omega}(\tau)$ *such that, with* $\Omega_N^{(1)}$ *and* $\mathcal{E}_{N,\omega}^{(1)}$ *as in Definition 9.11,*

(1)

$$\begin{aligned} &\mathrm{mes}\,(\mathcal{B}_{N,\omega}) \le \exp(-(\log N)^A), \;\; \mathrm{compl}(\mathcal{B}_{N,\omega}) < N^9 \\ &\mathrm{mes}\,(\mathcal{B}'_{N,\omega}) \le \exp(-(\log\log N)^A), \;\; \mathrm{compl}(\mathcal{B}'_{N,\omega}) < \exp((\log\log N)^{A/2}) \end{aligned} \tag{10.8}$$

as well as

$$\begin{aligned} &\mathrm{mes}\,(\mathcal{E}_{N,\omega}(\tau)) \le \exp(-(\log\tau^{-1})(\log N)^{-C}), \;\; \mathrm{compl}(\mathcal{E}_{N,\omega}(\tau)) \le N^C \\ &\mathrm{mes}\,(\mathcal{B}_{N,\omega}(\tau)) \le \exp(-(\log\tau^{-1})(\log N)^{-C}), \;\; \mathrm{compl}(\mathcal{B}_{N,\omega}(\tau)) \le N^C \end{aligned} \tag{10.9}$$

(2) *If for some* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(1)}$, *and* $x \in \mathbb{T}$ *one has*

$$\mathrm{spec}\,\big(H_{[-N,N]}(x,\omega)\big) \cap \big((E',E'') \setminus (\mathcal{E}_{N,\omega}^{(1)} \cup \mathcal{E}_{N,\omega}(\tau))\big) \ne \emptyset,$$

then $x \in \mathbb{T} \setminus (\mathcal{B}_{N,\omega} \cup \mathcal{B}'_{N,\omega} \cup \mathcal{B}_{N,\omega}(\tau))$

(3) *For* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$ *one has*

$$|E_{j_1}^{(N)}(x,\omega) - E_{j_2}^{(N)}(x,\omega)| \ge \exp(-N^\delta) \tag{10.10}$$

for any $j_1 \ne j_2$, *provided* $E_{j_s}^{(N)}(x,\omega) \in (E',E'') \setminus \mathcal{E}_{N,\omega}$, $s = 1,2$. *For* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(1)}$ *one has*

$$|E_{j_1}^{(N)}(x,\omega) - E_{j_2}^{(N)}(x,\omega)| \ge \exp(-(\log N)^A) \tag{10.11}$$

for any $j_1 \ne j_2$, *provided* $E_{j_s}^{(N)}(x,\omega) \in (E',E'') \setminus \mathcal{E}_{N,\omega}^{(1)}$, $s = 1,2$

(4) *For* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(1)}$ *one has* $|\partial_x E_j^{(N)}(x,\omega)| \geq \tau$ *, provided*

$$E_j^{(N)}(x,\omega) \in (E', E'') \setminus (\mathcal{E}_{N,\omega}^{(1)} \cup \mathcal{E}_{N,\omega}(\tau))$$

Proof. In this proof we do not distinguish between $\tilde{H}_{[-N,N]}$ and $H_{[-N,N]}$. This is justified by the results of this section which demonstrate that the small "fattening" parameter σ_N can be ignored. Set

$$\mathcal{B}_{N,\omega} := \{x \in \mathbb{T} : \operatorname{spec}\big(H_N(x,\omega)\big) \cap \mathcal{E}_{N,\omega} \neq \emptyset\}$$
$$\mathcal{B}'_{N,\omega} := \{x \in \mathbb{T} : \operatorname{spec}\big(H_N(x,\omega)\big) \cap \mathcal{E}'_{N,\omega} \neq \emptyset\}$$

where $\mathcal{E}_{N,\omega}^{(1)} = \mathcal{E}_{N,\omega} \cup \mathcal{E}'_{N,\omega}$ as in Definition 9.11. Then conditions (10.8) hold for $\mathcal{B}_{N,\omega}$, $\mathcal{B}'_{N,\omega}$ due to Lemma 2.17 (the analogue of Wegner's estimate). Using the notations of Lemma 10.10 set

$$\mathcal{B}_{N,\omega}(\tau) := \bigcup_{j,m} (\overline{E}_j(2\tau,m), \underline{E}_j(2\tau,m)), \quad \mathcal{B}_{N,\omega}(\tau) := \bigcup_{j,m} (\eta'_{j,m}(2\tau), \eta''_{j,m}(2\tau)).$$

Then conditions (10.9) follow from Lemma 10.10. Property (2) holds due to the definition of the sets involved, whereas (3) is due to Proposition 10.1. Finally, property (4) is due to Lemma 10.9. ☐

Remark 10.12. *For future reference we note that in addition to* (1)–(4) *of Proposition 10.11, the following property holds due to Corollary 9.10: For any* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(1)}$ *and any* $x \in \mathbb{T}$*, any* ℓ^2*-normalized eigenfunction* $\psi_j^{(N)}(x,\omega)$ *of* $H_{[-N,N]}(x,\omega)$ *with associated eigenvalue* $E_j^{(N)}(x,\omega) \in \mathbb{R} \setminus \mathcal{E}_{N,\omega}^{(1)}$ *satisfies*

$$|\psi_j^{(N)}(x,\omega)(n)| \leq C \exp(-\gamma \operatorname{dist}(n, \Lambda_j)/2)$$

for all $n \in [-N,N]$ *where* $\Lambda_j := [\nu_j^{(N)}(x,\omega) - \ell, \nu_j^{(N)}(x,\omega) + \ell] \cap [-N,N]$ *with* $\ell = (\log N)^{4A}$.

11. Segments of eigenvalue parametrizations and their translations

In this section we discuss the segments of functions $E_j^{(N)}(\cdot,\omega)$ and establish a self-similar structure of these functions with regard to the shift by ω. The later property is based on the finite volume localization of Section 6, and should be considered as a precursor to the co-variant property of Sinai's function from Section 13. Let $\mathcal{B}_{N,\omega}, \mathcal{B}_{N,\omega}(\tau)$ be as in Proposition 10.11. Set $\tau_N = \exp(-(\log N)^B)$, $B \gg A$, and

$$\bar{\mathcal{B}}_{N,\omega} = \mathcal{B}_{N,\omega}^{(1)} \cup \mathcal{B}_{N,\omega}(\tau_N), \qquad \mathcal{B}_{N,\omega}^{(1)} := \mathcal{B}_{N,\omega} \cup \mathcal{B}'_{N,\omega}$$

There exist intervals $[\xi'_k, \xi''_k], \xi''_k < \xi'_{k+1},\ k = 1,\dots,k_0,\ k_0 \leq N^C$ such that

$$\mathbb{T} \setminus \bar{\mathcal{B}}_{N,\omega} = \bigcup_{1 \leq k \leq k_0} [\xi'_k, \xi''_k]$$

Set

$$\mathcal{B}''_{N,\omega} := \mathbb{T} \setminus \bigcup \big\{[\xi'_k, \xi''_k] \,:\, \xi''_k - \xi'_k > \exp(-(\log N)^{2A})\big\}$$

To point here is that we add all very short "good intervals" into the bad set. This does not increase the measure of the bad set too much. Indeed,

$$\operatorname{mes}(\mathcal{B}''_{N,\omega}) \leq \exp(-(\log N)^A), \ \operatorname{compl}(\mathcal{B}''_{N,\omega}) \leq N^C$$

We denote by $[\underline{x}_k, \bar{x}_k],\ k = 1,2,\dots,k_1,\ \bar{x}_k < \underline{x}_{k+1}$ the maximal intervals of $\mathbb{T} \setminus \mathcal{B}''_{N,\omega}$. Recall that due to Proposition 10.11

$$|\partial_x E_j^{(N)}(x,\omega)| > \tau_N \quad \text{for any} \quad x \in \bigcup_{1 \leq k \leq k_1} [\underline{x}_k, \bar{x}_k] \tag{11.1}$$

We summarize the properties of the intervals $[\underline{x}_k, \bar{x}_k]$ (including those mentioned in Remark 10.12) in the following lemma.

Lemma 11.1. *Let* $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N^{(1)}$*. There exists intervals* $[\underline{x}_k, \bar{x}_k]$ *(depending on* ω*),* $k = 1,\dots,k_1$*,* $\underline{x}_k < \bar{x}_{k+1}$ *such that*

(1) $|\partial_x E_j^{(N)}(x,\omega)| \geq \exp(-(\log N)^B)$ *for any* $x \in \bigcup_{1 \leq k \leq k_1} [\underline{x}_k, \bar{x}_k]$ *and any* $1 \leq j \leq 2N+1$

(2) *for any* $x \in \bigcup_{1\le k\le k_1} [\underline{x}_k, \bar{x}_k]$ *and any* j *there exists* $\nu_j^{(N)}(x,\omega) \in [-N,N]$ *such that*

$$|\psi_j^{(N)}(x,\omega,n)| \le \exp\big(-\frac{\gamma}{2}|n-\nu_j^{(N)}(x,\omega)|\big)$$

provided $|n-\nu_j^{(N)}(x,\omega)| > N^{\frac12}$

(3) $\mathrm{mes}\,\big(\mathbb{T}\backslash \bigcup_{1\le k\le k_1} [\underline{x}_k, \bar{x}_x]\big) \le 3\exp(-(\log N)^A)$

(4) $k_1 \le N^C$

(5) $\bar{x}_k - \underline{x}_k \ge \exp(-(\log N)^{2A}),\ \ k = 1,2,\dots,k_1$

Definition 11.2. *We refer to each triplet* $\{E_j^{(N)}(x,\omega), \underline{x}_k, \bar{x}_k\}$, $j = 1,\dots,2N+1$, $k = 1,\dots,k_1$ *as a segment of Rellich's parametrization, or just a segment of* $E_j^{(N)}$. *If* $\partial_x E_j^{(N)}(x,\omega) > 0$, *respectively* $\partial_x E_j^{(N)}(x,\omega) < 0$, *for any* $x \in [\underline{x}_k, \bar{x}_k]$ *then* $\{E_j^{(N)}(x,\omega), \underline{x}_k, \bar{x}_k\}$ *is called a positive–slope, respectively negative–slope, segment. We also refer to these segments as* I*-segments provided the open interval* $I \subset \mathbb{R}$ *is the range of the segment, i.e.,* $I = \{E_j^{(N)}(x,\omega) \,:\, \underline{x}_k < x < \bar{x}_k\}$.

Remark 11.3. *Let* $\{E_j^{(N)}(x,\omega), \underline{x}_k, \bar{x}_k\}$ *be a segment. Recall that due to Proposition 10.11 one has in particular*

$$|E_j^{(N)}(x,\omega) - E_{j_1}^{(N)}(x,\omega)| \ge \exp(-N^\delta)$$

for any $j_1 \ne j$. *Since* V *is analytic one infers from this and standard perturbation theory of Hermitian matrices that the function* $E_j^{(N)}(\cdot,\cdot)$ *admits an analytic continuation to the polydisk*

$$\mathcal{P} := \{(z,w) \in \mathbb{C}^2 \,:\, |z-e(x)| < r_0, |w-\omega| < r_0\}$$

where $r_0 := C(V)^{-1}\exp(-N^\delta)$. *Moreover,*

$$\sup_{\mathcal{P}} |E_j^{(N)}(z,w)| \le C(V)$$

Note that the same bound holds with the stronger estimate (10.11) *instead of the weaker one* (10.10), *albeit with* $\delta > 0$ *arbitrarily small (simply because of the stronger bound). For many applications below the weaker bound suffices, and we have chosen to use it for the remainder of the section. The reader will have no difficulty replacing it with the stronger one* (10.11) *whenever the need arises.*

We now turn to translations of the segments. For this purpose we are going to impose the condition

$$-N + N^{\frac12} \le \nu_j^{(N)}(x,\omega) \le N - N^{\frac12} \tag{11.2}$$

which guarantees that the window of localization is separated from the boundary of the box $[-N,N]$.

Lemma 11.4. *Using the notations of Proposition 7.1 assume that*

$$\mathrm{dist}\big(E_j^{(N)}(x,\omega), \mathcal{E}_{N,\omega}\big) > 2\exp\big(-N^\delta\big), \quad -N + N^{1/2} < \nu_j^{(N)}(x,\omega) < N - N^{1/2}$$

Then for any k *such that* $-N + N^{1/2}/2 < \nu_j^{(N)}(x,\omega) + k < N - N^{1/2}/2$ *there exists a unique* $E_{j_k}^{(N)}(x + k\omega,\omega) \in \mathrm{spec}\,\big(H_{[-N,N]}(x+k\omega,\omega)\big)$ *such that*

$$\big|E_j^{(N)}(x,\omega) - E_{j_k}^{(N)}(x+k\omega,\omega)\big| < \exp\big(-\gamma_2 N^{1/2}\big)\ , \tag{11.3}$$

$$\big|\partial_x E_j^{(N)}(x,\omega) - \partial_x E_{j_k}^{(N)}(x+k\omega,\omega)\big| < \exp\big(-\gamma_2 N^{1/2}\big)\ , \tag{11.4}$$

as well as

$$E_{j_k}^{(N)}(x+k\omega,\omega) \notin \mathcal{E}_{N,\omega} \ , \tag{11.5}$$

$$\big|\nu_{j_k}^{(N)}(x+k\omega,\omega) - \big(\nu_j^{(N)}(x,\omega)+k\big)\big| \le N^{1/2}/4 \ , \tag{11.6}$$

$$-N+N^{1/2}/4 < \nu_{j_k}^{(N)}(x+k\omega,\omega) < N - N^{1/2}/4 \ , \tag{11.7}$$

$$\sum_{|m+k-\nu_j^{(N)}(x,\omega)|\le N^{1/2}/4} \big|\psi_{j_k}^{(N)}(x+k\omega,m) - \psi_j^{(N)}(x,m+k)\big|^2 < \exp\big(-\gamma_3 N^{1/2}\big) \tag{11.8}$$

where $\gamma_t = 2^{-t+1}\gamma_1$.

Proof. Note that

$$\begin{aligned} H_{[-N,N]}(x+k\omega,\omega)\big(\psi_j^{(N)}(x,\omega,\cdot+k)\big)(m) &= H_{[-N,N]}(x,\omega)\big(\psi_j^{(N)}(x,\omega.)\big)(m+k) \\ &= E_j^{(N)}(x,\omega)\psi_j^{(N)}(x,\omega,k+m) \end{aligned} \tag{11.9}$$

provided $-N < m+k < N$ and $-N < m < N$. Recall also that $\big|\psi_j^{(N)}(x,\omega,\pm N)\big| \le \exp\big(-\gamma_3 N^{1/2}\big)$, since $-N+N^{1/2} < \nu_j^{(N)}(x,\omega) < N-N^{1/2}$ Hence

$$\Big\|\big(H_{[-N,N]}(x+k\omega,\omega) - E_j^{(N)}(x,\omega)\big)\psi_j^{(N)}(x,\omega,\cdot+k)\Big\| < \exp\big(-\gamma_4 N^{1/2}\big) \ . \tag{11.10}$$

Therefore, there exists $E_{j_k}^{(N)}(x+k\omega,\omega) \in \big(E_j^{(N)}(x,\omega) - \exp\big(-\gamma_5 N^{1/2}\big), E_j^{(N)}(x,\omega) + \exp\big(-\gamma_5 N^{1/2}\big)\big)$. Moreover, due to our assumptions on $E_j^{(N)}(x,\omega)$, one has $E_{j_k}^{(N)}(x+k\omega,\omega) \notin \mathcal{E}_{N,\omega}$. Hence,

$$\big|E_{j_k}^{(N)}(x+k\omega,\omega) - E_{j'}^{(N)}(x+k\omega,\omega)\big| > \exp(-N^\delta) \tag{11.11}$$

for any $j' \ne j_k$. Relations (11.9)–(11.10) combined imply (11.3). Relations (11.5), (11.6) follow from (11.3). The estimate (11.8) follows from (11.10) and (11.11) via the spectral theorem for Hermitian matrices. Finally, (11.4) follows from the well-known "Feynman formula" (or first order eigenvalue perturbation formula)

$$\partial_x E_j^{(N)}(x,\omega) = \sum_{\ell=-N}^{N} V'(x+\ell\omega)\big|\psi_j^{(N)}(x,\omega,\ell)\big|^2$$

and the preceding estimates. □

We now illustrate how to relate the localized eigenfunctions of consecutive scales.

Lemma 11.5. *Using the notations of Proposition 7.1 assume that* $\omega \in \mathbb{T}_{c,a} \setminus \big(\Omega_N \cup \Omega_{N'}\big)$, *where* $N' \asymp \exp\big((\log\log N)^{C_1}\big)$, $C_1 \gg C$, *and with* $Q = \exp\big((\log\log N)^C\big)$. *If*

$$E_j^{(N)}(x,\omega) \notin \mathcal{E}_{N,\omega}, \qquad \operatorname{dist}\big(E_j^{(N)}(x,\omega), \mathcal{E}_{N',\omega}\big) > \exp\big(-(N')^{1/2}\big),$$

then there exists $\nu \in \mathbb{Z}$, $|\nu - \nu_j^{(N)}(x,\omega)| \le Q$ *and*

$$E_{j'}^{(N')}(x+\nu\omega,\omega) \in \big(E_j^{(N)}(x,\omega) - \exp(-\gamma_1 N'), E_j^{(N)}(x,\omega) + \exp(-\gamma_1 N')\big) \ ,$$

where $\gamma_1 = c\gamma$, $\gamma = \inf L(E,\omega)$. *Moreover, the corresponding normalized eigenfunctions*

$$\psi_j^{(N)}(x,\omega,k), \qquad \psi_{j'}^{(N')}(x+\nu\omega,\omega,k-\nu)$$

satisfy

$$\sum_{k\in[\nu-N',\nu+N']} \big|\psi_j^{(N)}(x,\omega,k) - \psi_{j'}^{(N')}(x+\nu\omega,\omega,k-\nu)\big|^2 \le \exp(-\gamma_1 N') \ . \tag{11.12}$$

Proof. Assume first $-N+N' < \nu_j^{(N)}(x,\omega) < N-N'$. Then with $\nu = \nu_j^{(N)}(x,\omega)$ one has:

$$\big\|\big(H_{[\nu-N',\nu+N']}(x,\omega) - E_j^{(N)}(x,\omega)\big)\psi_j^{(N)}(x,\omega,\cdot)\big\| \le \exp(-\gamma N'/4)\ , \tag{11.13}$$

$$1 - \sum_{k\in[\nu-N',\nu+N']} \big|\psi_j^{(N)}(x,\omega,k)\big|^2 < \exp(-\gamma N'/4) \tag{11.14}$$

due to Proposition 7.1. Hence, there exists

$$E_{j'}^{(N')}(x+\nu\omega,\omega) \in \big(E_j^{(N)}(x,\omega) - \exp(-\gamma_1 N'), E_j^{(N)}(x,\omega) + \exp(-\gamma_1 N')\big).$$

Moreover, due to assumptions on $E_j^{(N)}(x,\omega)$, one has $E_{j'}^{(N')}(x+\nu\omega,\omega) \notin \mathcal{E}_{N',\omega}$. Hence,

$$\big|E_{j'}^{(N')}(x+\nu\omega,\omega) - E_k^{(N')}(x+\nu\omega,\omega)\big| > \exp\big(-(N')^\delta\big) \tag{11.15}$$

for any $k \ne j'$. Then (11.13)–(11.15) combined imply (11.12) (expand in the orthonormal basis $\{\psi_k^{(N')}\}_k$). If $\nu_j^{(N)}(x,\omega) \le -N+N'$ (resp., $\nu_j^{(N)}(x,\omega) \ge N-N'$), then (11.13)–(11.15) are valid with $\nu_j^{(N)}(x,\omega) = -N+N'$ (resp., with $\nu_j^{(N)}(x,\omega) = N-N'$). □

Recall that $\nu_j^{(N)}(x,\omega)$ is stable under perturbations of $H_{[-N,N]}(x,\omega)$ of magnitude $\exp(-(\log N)^C)$. Since some of the interval $[\underline{x}_k,\bar{x}_k]$ are definitely of a larger size, condition (11.2) can hold on some part of $[\underline{x}_k,\bar{x}_k]$ and fail on another one. For that reason we consider also all the triplets

$$\big\{E_j^{(N)}(x), \underline{x}, \bar{x}\big\}$$

with $[\underline{x},\bar{x}]$ being an arbitrary subsegment of some $[\underline{x}_1,\bar{x}_1]$, $k=1,2,\dots,k_1$.

Definition 11.6. *A segment* $\{E_j^{(N)}(x,\omega),\underline{x},\bar{x}\}$ *is called* regular *if the condition*

$$-N+N^{\frac12} \le \nu_j^{(N)}(x,\omega) \le N-N^{\frac12} \tag{11.16}$$

holds for every $x\in[\underline{x},\bar{x}]$.

The condition (11.16), which requires the associated eigenfunction to be properly localized inside of the base interval $[-N,N]$, will play a central role in this paper. It ensures *stability* of resonances as one passes from one scale to the next.

12. Formation of Regular Spectral Segments

The section is devoted to the comparison of the zeros in energy of each entry of $M_N(e(x),\omega,E_0)$. Recall that these entries are are determinants $f_{[a,N-b]}(e(x),\omega,E)$ where $a,b=1,2$. The main theme will be to single out a "good case" characterized by each determinant having a zero very close to E_0 (we refer to E_0 as a *unconditional spectral value at scale* N in that case, see Definition 12.2 below). The significance of this idea lies with the induction in the scale; indeed, in passing from $[1,N]$ to $[-\bar{N},\bar{N}]$ with $\bar{N}=N^C$ we shall see that if E_0 is an unconditional spectral value at scale N, then it remains very close to the spectrum at the larger scale $\bar{N}$. Furthermore, this will be the crucial vehicle for constructing *regular* segments as introduced in Definition 11.6, see Proposition 12.6 below which is the main result of this section. With the notations of Section 10, define

$$\tilde{\Omega}_N := \Omega_N \cup \Omega_{N-1} \cup \Omega_{N-2}$$
$$\tilde{\mathcal{B}}_{N,\omega} := \mathcal{B}_{N,\omega} \cup \mathcal{B}_{N-1,\omega} \cup \mathcal{B}_{N-2,\omega}$$
$$\tilde{\Omega}_N^{(1)} := \Omega_N^{(1)} \cup \Omega_{N-1}^{(1)} \cup \Omega_{N-2}^{(1)}$$
$$\tilde{\mathcal{B}}_{N,\omega}^{(1)} := \mathcal{B}_{N,\omega}^{(1)} \cup \mathcal{B}_{N-1,\omega}^{(1)} \cup \mathcal{B}_{N-2,\omega}^{(1)}$$

where Ω_N, $\mathcal{B}_{N,\omega}$, $\Omega_N^{(1)}$, $\mathcal{B}_{N,\omega}^{(1)}$ are the same as in Proposition 10.11 and $E_j^{(N)}$ be as in the previous section. In the following lemma, we begin with the comparison of the spectra of the entries, as indicated in the previous paragraph.

Lemma 12.1. *Using the notation from above, one has the following:*

(1) *Let $\omega \in \mathbb{T}_{c,a} \setminus \Omega_N$, and $x_0 \in \mathbb{T} \setminus \tilde{\mathcal{B}}_{N,\omega}$. Then*

$$\min\left[\operatorname{dist}(\operatorname{spec}\left(H_{[1,N-1]}(x_0,\omega)\right), E_{j_0}^{(N)}(x_0,\omega)), \operatorname{dist}(\operatorname{spec}\left(H_{[2,N]}(x_0,\omega)\right), E_{j_0}^{(N)}(x_0,\omega))\right] \le r_0 = \exp(-N(\log N)^{-C_0})$$

provided N is large.

(2) *Furthermore, let $r_0 = e^{-N^\delta}$ where $0 < \delta \ll 1$ is arbitrary but fixed and assume that $N \ge N_0(V,a,c,\gamma,\delta)$. If*

$$\max\left[\operatorname{dist}(\operatorname{spec}\left(H_{[1,N-1]}(x_0,\omega)\right), E_{j_0}^{(N)}(x_0,\omega)), \operatorname{dist}(\operatorname{spec}\left(H_{[2,N]}(x_0,\omega)\right), E_{j_0}^{(N)}(x_0,\omega))\right] \le r_0, \tag{12.1}$$

then

$$\operatorname{dist}(\operatorname{spec}\left(H_{[2,N-1]}(x_0,\omega)\right), E_{j_0}^{(N)}(x_0,\omega)) \le r_1 \tag{12.2}$$

where $r_1 = e^{-N^{\delta/2}}$.

(3) *Finally, assume that $\omega \in \mathbb{T}_{c,a} \setminus \tilde{\Omega}_N^{(1)}$ and $x_0 \in \mathbb{T} \setminus \tilde{\mathcal{B}}_{N,\omega}^{(1)}$. Let $r_0 = \exp(-(\log N)^A)$ with some constant A. Then (12.1) implies (12.2) with $r_1 = e^{-(\log N)^{A/2}}$ for $N \ge N_0(V,c,a,A)$.*

Proof. Let $E_0 = E_j^{(N)}(x_0,\omega)$. Then $f_N(e(x_0),\omega,E_0) = f_{[1,N]}(e(x_0),\omega,E_0) = 0$. Since

$$-f_{[1,N]}(e(x_0),\omega,E_0)f_{[2,N-1]}(e(x_0),\omega,E_0) + f_{[1,N-1]}(e(x_0),\omega,E_0)f_{[2,N]}(e(x_0),\omega,E_0) = 1 \tag{12.3}$$

one obtains

$$f_{[1,N-1]}(e(x_0),\omega,E_0)f_{[2,N]}(e(x_0),\omega,E_0) = 1$$

In particular,

$$\min(|f_{[1,N-1]}(e(x_0),\omega,E_0)|, |f_{[2,N]}(e(x_0),\omega,E_0)|) \le 1$$

Assume, for instance, that

$$|f_{[1,N-1]}(e(x_0),\omega,E_0)| \le 1$$

Then, due to Corollary 2.20 with $\eta = \exp(-N(\log N)^{-C_0})$, one has

$$(E_0 - \eta, E_0 + \eta) \cap \operatorname{spec}\left(H_{[1,N-1]}(x_0,\omega)\right) \ne \emptyset \tag{12.4}$$

This proves (1). Assume now that in addition to (12.4) one has

$$(E_0 - r_0, E_0 + r_0) \cap \operatorname{spec}\left(H_{[2,N]}(x_0,\omega)\right) \ne \emptyset \tag{12.5}$$

where $r_0 = e^{-N^\delta}$ where $0 < \delta$ is small and fixed. Let $E_j^{[a,N-b]}(x,\omega)$, $j = 1,2,\dots$ stand for the eigenvalues of $H_{[a,N-b]}(x,\omega)$, $a = 1,2$, $b = 0,1$. Due to (12.4) and (12.5), there exist $E_{j_1}^{[1,N-1]}(x_0,\omega)$, $E_{j_2}^{[2,N]}(x_0,\omega)$ such that

$$|E_{j_1}^{[1,N-1]}(x_0,\omega) - E_0|,\ |E_{j_2}^{[2,N]}(x_0,\omega) - E_0| < r_0$$

Due to Corollary 7.3, one now has

$$f_{[1,N]}(e(x_0),\omega,E) = (E - E_{j_0}^{[1,N]}(x_0,\omega))\chi_0(e(x_0),\omega,E) \tag{12.6}$$

$$f_{[1,N-1]}(e(x_0),\omega,E) = (E - E_{j_1}^{[1,N-1]}(x_0,\omega))\chi_1(e(x_0),\omega,E) \tag{12.7}$$

$$f_{[2,N]}(e(x_0,\omega,E) = (E - E_{j_2}^{[2,N]}(x_0,\omega))\chi_2(e(x_0),\omega,E) \tag{12.8}$$

where $\chi_k(z,\omega,E)$ is analytic in $\mathcal{D}(e(x_0),r_2)\times\mathcal{D}(E_0,r_2)$, $r_2 \asymp \exp(-N^{\delta/4})$, with ω being fixed, $\chi_k(z,\omega,E) \ne 0$ for any $(z,E) \in \mathcal{D}(e(x_0),r_2)\times\mathcal{D}(E_0,r_2)$, $k = 0,1,2$. Moreover,

$$NL(E_0,\omega) - N^{\delta/3} < \log|\chi_k(z,\omega,E)| < NL(E_0,\omega) + N^{\delta/3} \tag{12.9}$$

for any $(z,E) \in \mathcal{D}(e(x_0),r_2/2)\times\mathcal{D}(E_0,r_2/2)$. It follows from (12.3) and (12.6) – (12.9) that

$$\begin{gathered}|f_{[2,N-1]}(e(x_0),\omega,E)| \le |E - E_{j_0}^{[1,N]}(x_0,\omega))|^{-1}\exp(-NL(E_0,\omega) + 2N^{\delta/3}) + \\ |\theta(E)|\,|E - E_{j_0}^{[1,N]}(x_0,\omega)|^{-1}\,|E - E_{j_1}^{[1,N-1]}(x_0,\omega)|\,|E - E_{j_2}^{[2,N]}(x_0,\omega)|\exp(NL(E_0,\omega))\end{gathered} \tag{12.10}$$

for any $|E - E_0| < r_2/2$, where

$$\theta(E) := e^{-NL(E_0,\omega)}\chi_1\chi_2/\chi_0$$

and satisfies the bound

$$\exp(-3N^{\delta/3}) \le |\theta(E)| \le \exp(3N^{\delta/3})$$

Clearly there exists $|E_1 - E_0| \le 2r_0$ such that

$$|E_1 - E_{j_0}^{[1,N]}(x_0,\omega)|^{-1}\, |E_1 - E_{j_1}^{[1,N-1]}(x_0,\omega)||E_1 - E_{j_2}^{[2,N]}(x_0,\omega)| \le 2r_0$$

So,

$$|f_{[2,N-1]}(e(x_0),\omega,E_1)| \le \exp(NL(E_0,\omega) - N^{\delta} + 10N^{\delta/3})$$

Due to Corollary 2.20, one has with $\eta = \exp(-N^{\delta/2})$,

$$(E_0-\eta, E_0+\eta) \cap \operatorname{spec}\big(H_{[2,N-1]}(x_0,\omega)\big) \ne \emptyset$$

and we are done with case (2). Case (3) is completely analogous. □

The following definition introduces the crucial notion of an *unconditional spectral value*.

Definition 12.2. *Using the notation of Lemma 12.1 assume that with $r_0 = e^{-N^{\delta}}$,*

$$\max\big[\operatorname{dist}(\operatorname{spec}\big(H_{[1,N-1]}(x_0,\omega)\big), E_{j_0}^{(N)}(x_0,\omega)), \operatorname{dist}(\operatorname{spec}\big(H_{[2,N]}(x_0,\omega)\big), E_{j_0}^{(N)}(x_0,\omega))\big] \le r_0 \tag{12.11}$$

*In this case one says that $E_0 := E_{j_0}^{(N)}(x_0,\omega)$ is an r_0-*unconditional spectral value *of $H_{[1,N]}(x_0,\omega)$. Let $\{E_{j_0}^{(N)}(x,\omega), \underline{x}, \bar{x}\}$ be a segment of a Rellich graph. Assume that for any $x \in [\underline{x}, \bar{x}]$, $E_{j_0}^{(N)}(x,\omega)$ is an r_0-unconditional spectral value of $H_{[1,N]}(x,\omega)$. We call this segment an r_0-*unconditional spectral segment *of the Hamiltonian $H_{[1,N]}(\cdot,\omega)$.*

Note that by (2) of Lemma 12.1, each entry has a zero in energy which is close to $E_{j_0}^{(N)}(x_0,\omega)$ in case the latter is an unconditional spectral value. The importance of the unconditional spectral values lies with the fact that they are stable with regard to the induction on scales procedure. In other words, the unconditional energies at a small scale will turn out to belong to the spectrum (or rather, be close to it) at the next larger scale. This process can also be reversed: we will show later that the conditional spectral values die out when we pass to the next larger scale. The corresponding analysis appears later in this section when we discuss property (NS) which stands for "non-spectral".

Corollary 12.3. *Under the assumptions of Lemma 12.1 one has the following: There exists $N_0 = N_0(V,c,a,\gamma,\delta)$, or respectively, $N_0 = N_0(V,c,a,\gamma,A)$ so that the following holds for all $N \ge N_0$: Assume that $E_0 = E_{j_0}^{(N)}(x_0,\omega)$ is an r_0-unconditional spectral value of $H_{[1,N]}(x_0,\omega)$. Then*

$$\log\|M_N(e(x_0),\omega,E_0)\| \le NL(E_0,\omega) - N^{\delta/2} \tag{12.12}$$

or, respectively,

$$\log\|M_N(e(x_0),\omega,E_0)\| \le NL(E_0,\omega) - (\log N)^{A/2} \tag{12.13}$$

Conversely, assume that (12.12) (*respectively,* (12.13)) *holds. Then E_0 is an r_0'-unconditional spectral value of $H_{[1,N]}(x_0,\omega)$, with $r_0' := \exp(-N^{\delta/3})$* (*respectively, $r_0' := \exp(-(\log N)^{A/3})$*).

Proof. Inspection of the proof of Lemma 12.1 establishes the direct implication. Assume that (12.12) holds. Then

$$\log|f_{[a,N-b]}(e(x_0),\omega,E)| \le NL(E_0,\omega) - N^{\delta/2}$$

for any $a = 1,2$, and $b = 0,1$. Due to Corollary 2.20, one has with $\eta = \exp(-N^{\delta/3})$,

$$(E_0-\eta, E_0+\eta) \cap \operatorname{spec}\big(H_{[a,N-b]}(x_0,\omega)\big) \ne \emptyset$$

for any $a = 1,2$, and $b = 0,1$. Due to Definition 12.2 this means that E_0 is an r_0'-unconditional spectral value for $H_{[1,N]}(x_0,\omega)$. The case of (12.13) is similar. □

Recall that for any $x \in \mathbb{T}$, $E \in \mathbb{R}$, and any integers N_i', N_i'', $i = 1, 2$ such that $N_1' < N_2' < N_2'' < N_1''$ one has

$$\big|\log\|M_{[N_1',N_1'']}(e(x),\omega,E)\| - \log\|M_{[N_2',N_2'']}(e(x),\omega,E)\|\big| \le C(V,E)\max(N_2'-N_1', N_1''-N_2'') \tag{12.14}$$

Combining (12.14) with Corollary 12.3 implies the following important *stability property* implied by the concept of unconditional spectral values. Note that this *fails* if at least one entry does not have a zero close to E_0.

Corollary 12.4. *Using the notations of Lemma 12.1 assume that E_0 is an r_0-unconditional spectral value of $H_{[1,N]}(\cdot,\omega)$. Then for any $-(\log N)^C < N' \le 1 < N \le N'' < N + (\log N)^C$ one has*

$$\mathrm{dist}(\mathrm{spec}\,\big(H_{[N',N'']}(x_0,\omega)\big), E_0) < r_0'$$

where $r_0' := \exp(-N^{\delta/4})$, respectively $r_0' := \exp(-(\log N)^{A/4})$.

To continue the analysis of unconditional spectral segments we will use the avalanche principle expansion as in Proposition 9.6 for the following logarithm

$$\log|f_{[-\overline{N},\overline{N}]}(e(x_0),\omega,E)| \tag{12.15}$$

where $N^2 \le \overline{N} \le N^{10}$, $E \in \mathcal{D}(E_0, r_0)$, and N, x_0, E_0 are as in Corollary 12.4. We arrange the expansion so that one of the intervals, say $\Lambda_{k_0} = [n_{k_0}, n_{k_0+1}]$, will obey the following condition:

$$-(\log N)^C < n_{k_0} < 1 < N < n_{k_0+1} < N + (\log N)^C \tag{12.16}$$

Due to Corollary 9.8, one can assume that n_1 etc. are adjusted at scale $\ell := (\log N)^{C/6}$ relative to $\mathcal{D}(e(x_0), r_1) \times \mathcal{D}(E_0, r_1)$, $r_1 := \exp(-(\log N)^A)$. Finally, one can assume that

$$N \le \min_k (n_{k+1} - n_k) \le N + (\log N)^C \tag{12.17}$$

Set

$$\bar{\mathcal{B}}_{N,\omega} := \bigcup_{N \le N' \le N + (\log N)^C} \tilde{\mathcal{B}}_{N',\omega}$$

$$\bar{\mathcal{B}}^{(1)}_{N,\omega} := \bar{\mathcal{B}}_{N,\omega} \cup \bigcup_{N \le N' \le N + (\log N)^C} \mathcal{B}^{(1)}_{N',\omega}$$

The notations of this paragraph will be used in the following lemma and proposition.

Lemma 12.5. *Using the notations of Corollary 12.4 assume that E_0 is an r_0-unconditional spectral value of $H_{[1,N]}(\cdot,\omega)$. Assume that $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}_{N,\omega}$ (respectively $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}^{(1)}_{N,\omega}$). Then for any $|x - x_0| \le C^{-1} r_0$ each entry of the matrix*

$$M_{[n_{k_0}, n_{k_0+1}]}(e(x), \omega, \cdot)$$

has exactly one zero in the disk $\mathcal{D}(E_0, r_0')$. Furthermore,

$$\nu_{f_{[-\overline{N},\overline{N}]}(e(x),\omega,\cdot)}(E_0, r_0'') \ge 1 \tag{12.18}$$

where $r_0'' = (r_0')^{1/4}$.

Proof. The first part of the statement follows from Corollary 12.4 and Proposition 10.11. Due to Proposition 9.6 one has

$$\mathcal{J}(\log|f_{[-\overline{N},\overline{N}]}(z_0,\omega_0,\cdot)|, E_0, r_1, r_2) \ge \mathcal{J}(\log|f_{[n_{k_0},n_{k_0+1}]}(z_0,\omega_0,\cdot)|, E_0, r_1, r_2) - O(\sqrt{r_1}) \tag{12.19}$$

for any $(r_0')^{1/2} \le r_1 \le (r_0')^{1/3}$, $r_2 = r_1/4$. Since $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}_{N,\omega}$ (respectively $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}^{(1)}_{N,\omega}$) one can pick r_1 so that $f_{[n_{k_0},n_{k_0+1}]}(z_0,\omega_0,\cdot)$ has no zeros in $\mathcal{D}(E_0, 2r_1) \setminus \mathcal{D}(E_0, r_1)$. Then

$$4r_1^2 r_2^{-2} \mathcal{J}(\log|f_{[n_{k_0},n_{k_0+1}]}(z_0,\omega_0,\cdot)|, E_0, r_1, r_2) \ge 1 \tag{12.20}$$

(12.19) combined with (12.20) imply (12.18) via Lemma 4.1. $\square$

Using the notations of Definition 12.2, assume that E_0 is an r_0-unconditional spectral value of $H_{[1,N]}(x_0, \omega)$. Assume also that $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}_{N,\omega}$ (respectively, $x_0 \in \mathbb{T} \setminus \bar{\mathcal{B}}^{(1)}_{N,\omega}$). Due to Lemma 11.1 there exists a segment $\{E^{(N)}_{j_0}(x, \omega), \underline{x}, \bar{x}\}$ such that the following conditions hold:

$$(i) \quad x_0 \in [\underline{x}, \bar{x}], \quad (ii) \quad |\underline{x} - \bar{x}| \ge \exp(-(\log r_0^{-1})^B), \quad (iii) \quad |\partial_x E^{(N)}_{j_0}| \ge \exp(-(\log r_0^{-1})^B)$$

where $1 \ll B$. Set $\underline{x}_1 := \max(x_0 - C^{-1} r_0, \underline{x})$, $\bar{x}_1 := \min(x_0 + C^{-1} r_0, \bar{x})$. The following proposition is the main statement concerning the formation of regular spectral segments.

Proposition 12.6. *Using the above notations assume for instance that* $\{E^{(N)}_{j_0}(x, \omega), \underline{x}, \bar{x}\}$ *is a positive slope segment. Then for any* $x_1 \in [\underline{x}_1, \bar{x}_1] \setminus \mathcal{B}_{\overline{N},\omega}$ *(respectively,* $x_1 \in [\underline{x}_1, \bar{x}_1] \setminus \mathcal{B}^{(1)}_{\overline{N},\omega}$ *) and any* $a \in [-\frac{1}{8}\overline{N}, \frac{7}{8}\overline{N}]$ *there exists a positive slope regular spectral segment* $\{E^{(\overline{N})}_{j'}(x, \omega), \underline{x}', \bar{x}'\}$ *of* $H_{[-\overline{N},\overline{N}]}(x, \omega)$ *such that the following conditions hold:*

1. $x_1 \in [\underline{x}', \bar{x}']$,
2. $|\underline{x}' - \bar{x}'| \ge \exp(-(\log r_0^{-1})^B)$,
3. $|E^{(\overline{N})}_{j'}(x, \omega) - E^{(N)}_{j_0}(x, \omega)| \le (r')_0^{1/2}$,
4. $|\partial_x E^{(\overline{N})}_{j'}| \ge \exp(-(\log r_0^{-1})^B)$,
5. $a - N \le \nu^{(\overline{N})}_j(x, \omega) \le a + 2N$.

Proof. Let $a = 0$. Due to Lemma 12.5 one has

$$\operatorname{dist}\left[\operatorname{spec}\left(H_{[-\overline{N},\overline{N}]}(x, \omega)\right), E^{(N')}_{j_0}(x, \omega)\right] \le r_0''$$

for any $x \in [\underline{x}_1, \bar{x}_1]$. Assume that $x_1 \in [\underline{x}_1, \bar{x}_1] \setminus \mathcal{B}_{\overline{N},\omega}$. Due to Lemma 11.1 there exists a segment $\{E^{(\overline{N})}_{j_1}(x, \omega), \underline{x}_1, \bar{x}_1\}$ such that conditions (1), (2), (3) hold. Condition (4) follows from (3) combined with (iii) and the Cauchy estimates for the derivatives. Condition (5) follows just from the definition of the center of localization and the first part in Lemma 12.5. For arbitrary $a \in [-\frac{3}{4}\overline{N}, \frac{3}{4}\overline{N}]$ the argument is similar. $\square$

We now turn to the investigation of conditional spectral values. For technical reasons we formulate this property in the following way which does not require E_0 to be a zero of the first entry of M_N; in fact, it will be convenient to also state this at scale ℓ rather than N. (NS) here stands for *non-spectral*, which refers to the fact that E_0 will be separated from the spectrum of $H_N(e(x), \omega)$ (up to shifting the edges) uniformly in x, see Proposition 12.10.

Definition 12.7. *Let* Ω_ℓ, $\mathcal{E}_{\omega,\ell}$ *be as in Lemma 6.2. Let* $\omega \in \mathbb{T}_{c,a} \setminus \bigcup_{m=\ell-2,\ell-1,\ell} \Omega_m$, $x_0 \in [0, 1]$, *and* $E_0 \in \mathbb{R} \setminus \mathcal{E}_{\omega,\ell}$. *By condition* (NS) *we mean that at least one of the Dirichlet determinants*

$$f_{[1,\ell]}(e(x_0), \omega, \cdot),\ f_{[1,\ell-1]}(e(x_0), \omega, \cdot),\ f_{[2,\ell]}(e(x_0), \omega, \cdot),\ f_{[2,\ell-1]}(e(x_0), \omega, \cdot)$$

has no zeros in $\mathcal{D}(E_0, r_0)$, $r_0 := \exp(-\ell^\delta)$, *where* $0 < \delta \ll 1$ *is a parameter.*

Similarly to Corollary 12.3 one now has the following statement.

Lemma 12.8. *Assume that condition* (NS) *holds. Then*

$$\ell L(E_0, \omega) - \ell^{2\delta} \le \log\|M_\ell(e(x_0), \omega, E)\| \tag{12.21}$$

for any $|E - E_0| < r_0/2$. *Conversely, assume that* (12.21) *holds for any* $|E - E_0| < r_0/2$. *Then condition* (NS) *holds with* r_0 *replaced by* $r_0' := \exp(-\ell^{4\delta})$.

Proof. If (12.21) fails for some $|E - E_0| < r_0/2$, then

$$\log|f_{[a,\ell-b]}(e(x_0), \omega, E)| \le \ell L(E_0, \omega) - \ell^{2\delta}$$

for any $a = 1, 2$, and $b = 0, 1$. Due to Corollary 2.20, one has with $\eta := r_0$

$$(E_0 - \eta, E_0 + \eta) \cap \operatorname{spec}\left(H_{[a,\ell-b]}(x_0, \omega)\right) \neq \emptyset$$

for any $a = 1, 2$, and $b = 0, 1$, contrary to the assumptions of the lemma. For the converse, let us assume that some determinant $f_{[a,\ell-b]}(e(x_0), \omega, \cdot)$ has a zero at E_1, where $|E_1 - E_0| \leq r_0'$. Just as in the proof of Lemma 12.1 it follows that

$$\log |f_{[a,\ell-b]}(e(x_0), \omega, \cdot)| < \ell L(E_0, \omega) - \ell^{3\delta}$$

Consequently, if each entry of $M_\ell(e(x_0), \omega, \cdot)$ were to exhibit such a zero, then

$$\ell L(E_0, \omega) - \ell^{2\delta} > \log \|M_\ell(e(x_0), \omega, E)\|$$

which is a contradiction. □

In the following corollary we show that (NS) has a natural stability property.

Corollary 12.9. *Assume condition (NS). Then for any $-\ell^\delta < \ell' \leq 1 < \ell \leq \ell'' < \ell + \ell^\delta$, at least one of the Dirichlet determinants*

$$f_{[\ell'+1,\ell'']}(e(x_0), \omega, \cdot),\ f_{[\ell'+1,\ell''-1]}(e(x_0), \omega, \cdot),\ f_{[\ell'+2,\ell'']}(e(x_0), \omega, \cdot),\ f_{[\ell'+2,\ell''-1]}(e(x_0), \omega, \cdot)$$

has no zeros in $\mathcal{D}(E_0, r_0')$, where $r_0' := \exp(-\ell^{4\delta})$.

Proof. For any x, E, and any integers N_i', N_i'', $i = 1, 2$ such that $N_1' < N_2' < N_2'' < N_1''$ one has

$$\left| \log \|M_{[N_1', N_1'']}(e(x), E)\| - \log \|M_{[N_2', N_2'']}(e(x), E)\| \right| \leq C(V, E) \max(N_2' - N_1', N_1'' - N_2'') \tag{12.22}$$

Hence, the estimate (12.21) is stable under such changes to the length of the monodromy matrix as long as the change is much smaller than $\ell^{2\delta}$. In particular, this holds with a change of size ℓ^δ as stated in the corollary. □

The following proposition proves that conditional spectral values die out when we pass from scale ℓ to a larger scale N. Clearly, this is of great importance as it ensures that our inductive procedure can be carried out with unconditional spectral values E_0 rather than conditional ones. The relevance of the unconditionality property here stems from Proposition 12.6 which shows that at larger scales eigenfunctions with eigenvalues close to E_0 are localized away from the edges of the underlying interval. If this were not so, then these eigenfunctions would not be stable when passing the next larger scale. We note one disadvantage of the following proposition: for each x it gives a choice of four determinants. We will subsequently see that *periodic* boundary conditions can be used uniformly for all x. This is the reason whey periodic boundary conditions appear in this paper at all.

Proposition 12.10. *Let E_0 be arbitrary. Assume there is an interval (x_0', x_0'') with $x_0'' - x_0' \geq \ell^{-\delta/3}$ such that for any $x_0 \in (x_0', x_0'')$ condition (NS) holds. Then for any $|E - E_0| \leq r_0/4$ and any $x \in \mathbb{T}$, and $N \geq \ell^2$ at least one of the Dirichlet determinants*

$$f_{[1,N]}(e(x), \omega, \cdot),\ f_{[1,N-1]}(e(x), \omega, \cdot),\ f_{[2,N]}(e(x), \omega, \cdot),\ f_{[2,N-1]}(e(x), \omega, \cdot)$$

has no zeros in $\mathcal{D}(E, r_0'')$, where $r_0'' := \exp(-N^{8\delta})$.

Proof. Let $|E - E_0| \leq r_0/4$ be arbitrary. Note that (NS) holds on the disk $\mathcal{D}(E, r_0/4)$ for any $x_0 \in (x_0', x_0'')$. Let $x \in \mathbb{T}$ be arbitrary. We invoke once again the avalanche principle expansion (9.3). Let $\ell^2 \leq N \leq \ell^{10}$ be arbitrary. By Corollary 12.9, one can pick $1 \leq a \leq \ell^\delta$, $N - \ell^\delta \leq b \leq N$, and arrange the avalanche principle expansion so that the following conditions hold:

(1) $[a, b] = \bigcup_{k=1}^{t} \Lambda_k$, $\Lambda_k = [n_k + 1, n_{k+1}]$
(2) for any $k = 1, \ldots, t-1$ there exists $|n_k' - n_k| \leq \ell^\delta$, such that $x + n_k'\omega \in (x_0', x_0'')$
(3) $|n_{k+1} - n_k - \ell| \leq 2\ell^\delta$
(4) $n_1, \ldots, n_t$ are adjusted to $\mathcal{D}(e(x), r_0/C(V)) \times \mathcal{D}(E, r_0/2)$, $r_0 = \exp(-\ell^\delta)$ at scale $\ell^{\delta/6}$

In item (4) we used Corollary 9.8, whereas for (2) we invoked the dynamics: since $\omega \in \mathbb{T}_{c,a}$, for any $x \in \mathbb{T}$ and any $s' \in \mathbb{Z}, s' > 0$ there exists $s' \le s \le s'^2$ such that

$$\|x - s\omega\| \le 1/s \tag{12.23}$$

In view of (1)–(4) we can apply the zero count of Proposition 9.6; the crucial observation here is that due to Corollary 12.9 one can pick the edge points n_k so that $f_{[n_k+1,n_{k+1}]}(e(x),\omega,\cdot)$ has no zeros in $\mathcal{D}(E,r_0')$, where $r_0' := \exp(-\ell^{4\delta})$. In conclusion,

$$\nu_{f_{[a,b]}(e(x),\omega,\cdot)}(E,r_0'/2) = \sum_{k=0}^{t-1} \nu_{f_{[n_k+1,n_{k+1}]}(e(x),\omega,\cdot)}(E_0,r_0'/2) = 0$$

This means that (NS) holds for the entries of $M_{[a,b]}(e(x),\omega,\cdot)$ on the disk $\mathcal{D}(E,r_0'/2)$ for all $x \in (x_0',x_0'')$. By Corollary 12.9 one can now replace $[a,b]$ by $[1,N]$ whence the proposition for the range $\ell^2 \le N \le \ell^{10}$. For arbitrary $N \ge \ell^2$ one can use an induction argument. Indeed, by the preceding any $x_0 \in (x_0',x_0'')$ has the property that (NS) holds for the entries of $M_{[1,N^{(1)}]}(e(x),\omega,\cdot)$ on the disk $\mathcal{D}(E,r^{(1)})$, with $N^{(1)} := \ell^8$ and $r^{(1)} := \exp\big(-(N^{(1)})^\delta\big)$. Therefore, one can apply the very same argument with $N^{(1)}$ in the role of ℓ. □

We now discuss the relation between the unconditional Dirichlet spectral values and the periodic spectrum. For this purpose, recall the following relation from Proposition 3.3 and 3.7 which follows from properties of the trace:

$$\log|g_N(e(x),\omega,E)| = \log\|M_{2N}(e(x),\omega,E)\| - \log\|M_N(e(x),\omega,E)\|] + O\big(\exp(-(\log N)^B)\big) \tag{12.24}$$

provided $\|N\omega\| \le \kappa_0(V,c,a,\gamma)$ and $E \in \mathbb{C}\setminus\mathcal{E}_x$, where $\mathcal{E}_x \subset \mathbb{C}$, $\mathrm{mes}\,(\mathcal{E}_x) \le \exp(-(\log N)^B)$. Here $g_N(e(x),\omega,E) := \det(H_N^{(P)}(x,\omega)-E)$ with $H_N^{(P)}(x,\omega)$ the Schrödinger operator on $[1,N]$ with periodic boundary conditions, and

$$g_N(e(x),\omega,E) = \mathrm{tr}\,(M_N(e(x),\omega,E)) - 2$$

The reason for considering periodic boundary conditions is as follows: note that Proposition 12.10 shows that under the (NS) condition at scale ℓ, for each $x \in \mathbb{T}$ it is possible to "wiggle" the boundary of $[1,N]$ slightly so as to ensure that the corresponding entry has no zeros in a disk of energies. The technically unpleasant feature here is that the "wiggling" or in precise terms, the choice of boundary conditions, depends on x. However, we will now see that (12.24) implies that periodic boundary conditions achieve the desired absence of zeros in E *uniformly* for all $x \in \mathbb{T}$.

Lemma 12.11. *Assume that for some x_0, E_0 and with $2N$ in the role of ℓ condition (NS) holds. Then*
(1) $\mathcal{J}\big(\log|g_N(e(x_0),\omega,\cdot)|,E_0,r\big) \le \exp(-(\log N)^C)$ *for any* $\exp(-N^{1/2}) \le r \le r_1 := \exp(-N^{10\delta})$
(2) $\nu_{g_N(e(x_0),\omega,\cdot)}(E,r_1/2) = 0$

Proof. Since each logarithm involved in (12.24) is subharmonic in E, one has

$$\mathcal{J}\big(\log|g_N(e(x_0),\omega,\cdot)|,E_0,r\big) \le \mathcal{J}\big(\log\|M_{2N}(e(x_0),\omega,\cdot)\|,E_0,r\big) + C\exp(-(\log N)^B)$$

Therefore, the estimate in (1) is due to Proposition 8.6. Part (2) follows from (1) due to Lemma 4.1. □

Property (2) in Lemma 12.11 simply says that the periodic spectrum does not intersect the interval (E_0',E_0''). We record this fact as a separate statement and definition.

Corollary 12.12. *Using the notations of Proposition 12.10 one has with* $(E_0',E_0'') := (E_0-r_0/8, E_0+r_0/8)$

$$\mathrm{spec}\,\big(H_N^{(P)}(x,\omega)\big) \cap (E_0',E_0'') = \emptyset \tag{12.25}$$

for any $x \in \mathbb{T}$ and any $N \ge \ell^2$ provided $\|N\omega\| \le \kappa_0(V,c,a,\gamma)$. An interval (E_0',E_0'') is called spectrum free *if there exists N_0 depending on the usual parameters a,c,V,ρ_0 as well as on (E_0',E_0'') such that* (12.25) *holds for any $N \ge N_0$ with $\|N\omega\| \le \kappa_0$ and any $x \in \mathbb{T}$.*

The following lemma establishes a crucial dichotomy between an interval being spectrum free and the existence of a regular segment.

Lemma 12.13. *Given a scale ℓ, a parameter $0 < \delta \ll 1$, and intervals (E_0', E_0''), (x_0', x_0'') with*
$$E_0'' - E_0' \ge \exp(-(\log \ell)^A), \quad x_0'' - x_0' \ge \ell^{-\delta/3}$$
either the interval
$$\big(E_0' + \frac{1}{4}\exp(-(\log \ell)^A),\ E_0'' - \frac{1}{4}\exp(-(\log \ell)^A)\big)$$
is spectrum free, or for any scale $\ell^2 \le N \le \ell^{10}$ and any $\frac{1}{4}N \le a \le \frac{3}{4}N$ there exists a regular I_1-segment
$$\big\{E_{j_1}^{(N)}(x,\omega), \underline{x}_1, \overline{x}_1\big\}, \quad I_1 \subset (E_0', E_0'')$$
with
$$(\underline{x}_1, \overline{x}_1) \subset (x_0', x_0''), \quad a - 2\ell \le \nu_j^{(N)}(x,\omega) \le a + 2\ell$$

Proof. Assume that there exist $x_0 \in (x_0', x_0'')$ and
$$E_0 \in \big(E_0' + \frac{1}{4}\exp(-(\log \ell)^A), E_0'' - \frac{1}{4}\exp(-(\log \ell)^A)\big)$$
such that condition (NS) fails. Then there exists j_1 such that
$$E_1 := E_{j_1}^{(\ell)}(x_0,\omega) \in \big(E_0' + \frac{1}{8}\exp(-(\log \ell)^A),\ E_0'' - \frac{1}{8}\exp(-(\log \ell)^A)\big)$$
is an r_0-unconditional spectral value, where $r_0 := \frac{1}{4}\exp(-\ell^\delta)$. Due to Proposition 12.6 for any $\ell^2 \le N \le \ell^{10}$ and any $\frac{1}{4}N \le a \le \frac{3}{4}N$ there exists a regular I_1-segment
$$\big\{E_{j_1}^{(N)}(x,\omega), \underline{x}_1, \overline{x}_1\big\}, \quad I_1 \subset (E_0', E_0'')$$
with
$$(\underline{x}_1, \overline{x}_1) \subset (x_0', x_0''), \quad a - 2\ell \le \nu_j^{(N)}(x,\omega) \le a + 2\ell$$
Fix arbitrary
$$E_0 \in \big(E_0' + \frac{1}{4}\exp(-(\log \ell)^A), E_0'' - \frac{1}{4}\exp(-(\log \ell)^A)\big) \tag{12.26}$$
and assume that for any $x_0 \in (x_0', x_0'')$ condition (NS) holds. There exists $N \asymp \ell^C$ such that $\|N\omega\| \le \kappa_0$ with κ_0 as in Proposition 3.3. Then due to Corollary 12.12 there exists a spectrum free subinterval $(E', E'') \subset (E_0', E_0'')$ containing E_0. Since E_0 as in (12.26) is arbitrary, the statement holds. □

To proceed, we need a version of Lemma 11.1 for the parametrization of the eigenvalues of the Schrödinger operator with periodic boundary conditions. Let
$$E_1^{(N,P)}(x,\omega) \le E_2^{(N,P)}(x,\omega) \le \cdots \le E_N^{(N,P)}(x,\omega)$$
be the eigenvalues of $H_{[1,N]}^{(P)}(x,\omega)$. We define segments of the graphs of $E_1^{(N,P)}(\cdot,\omega)$ via approximation by the segments $\big\{E_{j_1}^{(2N)}(x,\omega), \underline{x}_1, \overline{x}_1\big\}$.

Lemma 12.14. *Assume that $\|N\omega\| \le \kappa_0$, with κ_0 as in Proposition 3.3. Then the following properties hold:*

(1) *Let $\mathcal{E}_{2N,\omega}^{(1)}$ be as in Proposition 10.11 with $2N$ in the role of N. Assume that for some x_0, j one has*
$$\operatorname{dist}(E_j^{(N,P)}(x_0,\omega), \mathcal{E}_{2N,\omega}^{(1)}) \ge 2\exp(-N^\delta).$$
Then there exists an unconditional segment $\big\{E_{j_1}^{(2N)}(x,\omega), \underline{x}_1, \overline{x}_1\big\}$ with $x_0 \in (\underline{x}_1, \overline{x}_1)$ such that
$$|E_j^{(N,P)}(x,\omega) - E_{j_1}^{(2N)}(x,\omega)| \le \exp(-N^{2\delta}) \tag{12.27}$$
for any $x \in [\underline{x}_1, \overline{x}_1]$

(2) *Let $\big\{E_{j_1}^{(2N)}(x,\omega), \underline{x}_1, \overline{x}_1\big\}$ be a regular segment such that $\frac{1}{4}N \le \nu_{j_1}^{(2N)}(\cdot,\omega) \le \frac{1}{2}N$ on that segment. Then there exists $E_j^{(N,P)}(\cdot,\omega)$ such that (12.27) holds for all $x \in (\underline{x}_1, \overline{x}_1)$.*

Proof. Due to Lemma 12.11 there exists j_1 such that (12.27) holds for $x = x_0$. Note that $E_{j_1}^{(2N)}(x,\omega) \notin \mathcal{E}_{2N,\omega}^{(1)}$. Hence there exists a segment $\{E_{j_1}^{(2N)}(x,\omega), \underline{x}_1, \overline{x}_1\}$, with $x_0 \in (\underline{x}_1, \overline{x}_1)$. Once again due to Lemma 12.11 for any x there exists $j(x)$ such that

$$|E_j^{(N,P)}(x,\omega) - E_{j(x)}^{(2N)}(x,\omega)| \le \exp(-N^{2\delta}) \tag{12.28}$$

Recall that for any $j \neq j_1$ and any $x \in [\underline{x}_1, \overline{x}_1]$ one has

$$|E_j^{(2N)}(x,\omega) - E_{j_1}^{(2N)}(x,\omega)| \ge \exp(-(\log N)^A) \tag{12.29}$$

Combining (12.29) with (12.27) for $x = x_0$ one concludes that $j(x) = j_1$ for any $x \in [\underline{x}_1, \overline{x}_1]$ in (12.28). Thus, (12.27) holds. It follows from Lemma 12.11 and (12.27) that each value $E_{j_1}^{(2N)}(x,\omega)$ is unconditional. That proves the first part. To establish the second part, let us note that since $\frac{1}{4}N \le \nu_{j_1}^{(2N)}(\cdot,\omega) \le \frac{1}{2}N$, one has

$$\|(H_{[1,N]}^{(P)}(x_1,\omega) - E_{j_1}^{(2N)}(x_1,\omega))\psi_{j_1}^{(2N)}(x_1,\omega,\cdot)\| \le \exp(-\gamma N/8)$$

for any $x_1 \in (\underline{x}_1, \overline{x}_1)$. Hence, for any x_1 there exists $j(x_1)$ such that

$$|E_{j(x_1)}^{(N,P)}(x_1,\omega) - E_{j_1}^{(2N)}(x_1,\omega)| \le \exp(-\gamma N/8)$$

Let $x_1 = (\underline{x}_1 + \overline{x}_1)/2$. Then due to the lower bound on the Dirichlet graphs, the last estimate implies in particular that

$$\operatorname{dist}(E_{j(x_1)}^{(N,P)}(x_1,\omega), \mathcal{E}_{2N,\omega}^{(1)}) \ge 2\exp(-N^\delta)$$

Due to first part there exists j_2 such that

$$|E_{j(x_1)}^{(N,P)}(x_1,\omega) - E_{j_2}^{(2N)}(x_1,\omega)| \le \exp(-N^{2\delta})$$

for any $x_1 \in (\underline{x}_1, \overline{x}_1)$. Due to the separation property of the segments one concludes that $j_2 = j_1$ and we are done. $\square$

The following proposition establishes a dichotomy which will be an essential ingredient in our proof of Theorem 1.2. It shows that either an interval does not intersect the infinite volume spectrum, or it has to contain the graphs of both a positive slope as well as of a negative slope *regular* segment at all sufficiently large scales. In view of Figure 2 this is of course important for the creation of resonances and thus also, gaps. As already mentioned before, the regularity of the segments is essential for the rigorous implementation of Figure 2.

Proposition 12.15. *Given ℓ large, a parameter $0 < \delta \ll 1$, and intervals (E_0', E_0''), (x_0', x_0'') with $E_0'' - E_0' \ge \exp(-(\log \ell)^A)$, $x_0'' - x_0' \ge \ell^{-\delta/3}$, there is the following dichotomy:*

- *either the interval $(E_0' + \frac{1}{4}\exp(-(\log \ell)^A), E_0'' - \frac{1}{4}\exp(-(\log \ell)^A))$ is spectrum free*
- *or at some scale $\ell^2 \le N \le \ell^{10}$ there exist a regular positive slope I-segment $\{E_{j_1}^{(N)}(x,\omega), \underline{x}_1, \overline{x}_1\}$, as well as a regular negative slope I-segment $\{E_{j_2}^{(N)}(x,\omega), \underline{x}_2, \overline{x}_2\}$ with $I \subset (E_0', E_0'')$ and $(\underline{x}_1, \overline{x}_1) \subset (x_0', x_0'')$, with the same I.*

Proof. Assume that the first alternative does not hold. Then, due to Lemma 12.13 one can assume that for any $\ell^2 \le N_1 \le \ell^4$ there exists a regular I_1-segment $\{E_{j_1}^{(2N_1)}(x,\omega), \underline{x}_1, \overline{x}_1\}$ with

$$I_1 \subset (E_0', E_0''), \quad (\underline{x}_1, \overline{x}_1) \subset (x_0', x_0''), \quad \frac{1}{4}N_1 \le \nu_{j_1}^{(2N_1)}(x,\omega) \le \frac{1}{2}N_1$$

Since $\omega \in \mathbb{T}_{c,a}$, one can choose N_1 so that $\|N_1\omega\| \le \kappa_0(V,c,a,\gamma)$. Then due to the part (2) of Lemma 12.14 there exists j such that

$$|E_j^{(N_1,P)}(x,\omega) - E_{j_1}^{(2N_1)}(x,\omega)| \le \exp(-N_1^{2\delta}) \tag{12.30}$$

for any $x \in (\underline{x}_1, \overline{x}_1)$. Assume for instance that $\{E_{j_1}^{(2N_1)}(x,\omega)\}$ is a positive-slope segment. Let

$$x_1 := (\underline{x}_1 + \overline{x}_1)/2, \qquad E_1 := E_{j_1}^{(2N_1)}(x_1,\omega)$$

Without loss of generality (because of the periodicity) we may assume that x_1 is close to the middle of $\mathbb{T} \simeq [0,1]$. Due to the 1-periodicity $E_{j_1}^{(N_1,P)}(0,\omega) = E_{j_1}^{(N_1,P)}(1,\omega)$. Hence, either $E_1 \le E_{j_1}^{(N_1,P)}(0,\omega)$, or $E_1 \ge E_{j_1}^{(N_1,P)}(1,\omega)$. Assume for instance that the latter case takes place. Let

$$x_2 := (x_1 + \overline{x}_1)/2, \quad E_2 := E_{j_1}^{(2N_1)}(x_2,\omega), \quad \overline{E}_1 := E_{j_1}^{(2N_1)}(\overline{x}_1,\omega)$$

Then $E_1 + \exp((-\log N_1)^{2A}) < E_2$, $E_2 + \exp((-\log N_1)^{2A}) < \overline{E}_1$. Hence,

$$E_{j_1}^{(N_1,P)}(1,\omega) + \exp((-\log N_1)^{3A}) < E_{j_1}^{(N_1,P)}(x_2,\omega)$$
$$E_{j_1}^{(N_1,P)}(x_2,\omega) + \exp((-\log N_1)^{3A}) < E_{j_1}^{(N_1,P)}(\overline{x}_1,\omega)$$

Since $E_{j_1}^{(N_1,P)}(\cdot,\omega)$ is continuous, there exists $x_0 \in (\overline{x}_1, 1)$ such that

(i) $E_{j_1}^{(N_1,P)}(x_0,\omega) = E_{j_1}^{(N_1,P)}(x_2,\omega)$
(ii) for any $x \in (\overline{x}_1, x_0)$ one has $E_{j_1}^{(N_1,P)}(x,\omega) > E_{j_1}^{(N_1,P)}(x_2,\omega)$

Note that x_0 was chosen to be the first point to the right of $\overline{x}_1$ where the graph of $E_{j_1}^{(N_1,P)}(\cdot,\omega)$ hits the level $E_{j_1}^{(N_1,P)}(x_2,\omega)$. Due to part (1) of Lemma 12.14 there exists an unconditional segment $\{E_{j_2}^{(2N_1)}(x,\omega), \underline{x}_2, \overline{x}_2\}$ with $x_0 \in (\underline{x}_2, \overline{x}_2)$ such that

$$|E_j^{(N_1,P)}(x,\omega) - E_{j_2}^{(2N_1)}(x,\omega)| \le \exp(-N_1^{2\delta}) \tag{12.31}$$

for any $x \in (\underline{x}_2, \overline{x}_2)$. Because of (ii) above this must be a negative-slope segment. To see this, assume that it is a positive slope segment. Note that since $|E_{j(x_1)}^{(2N_1)}(x_0,\omega) - E_2| \le \exp(-N_1^{\delta})$ one has

$$\operatorname{dist}(E_{j(x_1)}^{(2N_1)}(x_0,\omega), \mathcal{E}_{2N_1,\omega}^{(1)}) \ge \exp(-(\log N_1)^{4A})$$

Therefore,

$$|\partial_x E_{j_2}^{(2N_1)}(x,\omega)| \ge \exp(-(\log N)^B)$$

for any $x \in [x_0^-, x_0^+]$, where $x_0^\pm := x_0 \pm \exp(-(\log N_1)^{5A})$. Due to part (1) of Lemma 12.14 one has

$$|E_j^{(N_1,P)}(x,\omega) - E_{j_2}^{(2N_1)}(x,\omega)| \le \exp(-N_1^{2\delta}) \tag{12.32}$$

for any $x \in [x_0^-, x_0^+]$. In particular, $E_j^{(N_1,P)}(x_0^-,\omega) < E_{j_1}^{(N_1,P)}(x_0,\omega) = E_{j_1}^{(N_1,P)}(x_2,\omega)$, which contradicts (ii) above. Thus, $\{E_{j_2}^{(2N_1)}(\cdot,\omega), \underline{x}_2, \overline{x}_2\}$ is indeed an unconditional, negative slope, segment. Applying Proposition 12.6 to $\{E_{j_1}^{(2N_1)}(\cdot,\omega), \underline{x}_1, \overline{x}_1\}$ and $\{E_{j_2}^{(2N_1)}(\cdot,\omega), \underline{x}_2, \overline{x}_2\}$ concludes the argument. The case $E_1 \le E_{j_1}^{(N_1,P)}(0,\omega)$ is treated analogously. □

13. The proof of Theorem 1.3

In this section we prove Theorem 1.3 on Sinai's parametrization of eigenfunctions. Sinai needed to assume cosine-like potentials and his argument was perturbative. Our construction applies to general analytic potentials under the condition that $L(\omega, E) > 0$. We derive Theorem 1.3 from the following detailed finite volume version.

Proposition 13.1. *Assume that $L(\omega_0, E) \ge \gamma > 0$ for some $\omega_0 \in \mathbb{T}_{c,a}$ and any $E \in (E', E'')$. Then there exist $\rho^{(0)} = \rho^{(0)}(V,c,a,\gamma) > 0$ and $N_0 = N_0(V,c,a,\gamma)$ such that for any $N \ge N_0$ there exists a subset $\Omega_N \subset \mathbb{T}$ so that for all $\omega \in \mathbb{T}_{c,a} \cap (\omega_0 - \rho^{(0)}, \omega_0 + \rho^{(0)}) \setminus \Omega_N$ there exist $\mathcal{E}_{N,\omega} \subset \mathbb{R}$, $\mathcal{B}_{N,\omega} \subset \mathbb{T}$ such that the following statements hold:*

(1) $\operatorname{mes}(\Omega_N) \le \exp(-(\log N)^{A_0})$, $\operatorname{compl}(\Omega_N) \le N^{C_0}$
$\operatorname{mes}(\mathcal{E}_{N,\omega}) \le \exp(-(\log N)^{A_0})$, $\operatorname{compl}(\mathcal{E}_{N,\omega}) \le N^{C_0}$
$\operatorname{mes}(\mathcal{B}_{N,\omega}) \le \exp(-(\log N)^{A_0})$, $\operatorname{compl}(\mathcal{B}_{N,\omega}) \le N^{C_0}$

(2) *For any* $x \in \mathbb{T}\backslash\mathcal{B}_{N,\omega}$ *one has*

$$\mathrm{spec}\,\big(H_{[-N,N]}(x,\omega)\big) \cap \big((E',E'') \cap \mathcal{E}_{N,\omega}\big) = \emptyset,$$

(3) *For any* $\overline{N} \geq N$ *one has*

$$\mathcal{S}_{\overline{N},\omega}\backslash\mathcal{S}_{N,\omega} \subset \mathcal{E}_{N,\omega}$$

where

$$\mathcal{S}_{N,\omega} := \bigcup_{x\in\mathbb{T}} \mathrm{spec}\,\big(H_{[-N,N]}(x,\omega)\big)$$

Furthermore, assume that for some $N \geq N_0$,

$$\omega \in \mathbb{T}_{c,a} \cap (\omega_0 - \rho^{(0)}, \omega^{(0)} + \rho^{(0)})\backslash \bigcup_{N'\geq N} \Omega_{N'}$$

Then the following further properties hold:

(4) *Let* $x \in \mathbb{T}\backslash\mathcal{B}_{N,\omega}$. *If some eigenvalue* $E_j^{(N)}(x,\omega)$ *falls into the interval* (E',E''), *then there exists* $\nu_j^{(N)}(x,\omega) \in [-N,N]$ *such that*

$$|\psi_j^{(N)}(x,\omega,n)| \leq \exp\big(-\frac{\gamma}{2}|n-\nu_j^{(N)}(x,\omega)|\big)$$

for all $|n-\nu_j^{(N)}(x,\omega)| \geq N^{1/2}$

(5) *Set* $\hat{\Omega}_N := \bigcup_{N'\geq N}\Omega_{N'}$, $\hat{\mathcal{B}}_{N,\omega} := \bigcup_{N'\geq N}\mathcal{B}_{N',\omega}$. *Let* $\omega \in \mathbb{T}_{c,a}\backslash\hat{\Omega}_N$ *and let* $x \in \mathbb{T}\backslash\hat{\mathcal{B}}_{N,\omega}$, $E_j^{(N)}(x,\omega) \in (E',E'')$. *Assume that*

$$\nu_j^{(N)}(x,\omega) \in [-N+N^{1/2}, N-N^{1/2}]$$

Then for each $N' \geq N$ *there exists* $j_{N'}$ *such that*

$$\begin{aligned}\big|E_{j_{N'}}^{(N')}(x,\omega) - E_j^{(N)}(x,\omega)\big| &< \exp(-\frac{\gamma}{2}N^{\frac12})\\ \big|\partial_x E_{j_{N'}}^{(N')}(x,\omega) - \partial_x E_j^{(N)}(x,\omega)\big| &< \exp(-\frac{\gamma}{2}N^{\frac12})\\ \big|\psi_{j_{N'}}^{(N')}(x,\omega,n)\big| &\leq \exp\big(-\frac{\gamma}{2}|n-\nu_j^{(N)}(x,\omega)|\big)\end{aligned} \tag{13.1}$$

for any $|n| \leq N'$, *and any* $N' \geq N$;

$$|E_{j_{N''}}^{(N'')}(x,\omega) - E_{j_{N'}}^{(N')}(x,\omega)| \leq \exp(-\frac{\gamma}{2}(N')^{\frac12})$$
$$|\psi_{j_{N''}}^{(N'')}(x,\omega,n) - \psi_{j_{N'}}^{(N')}(x,\omega,n)| \leq \exp(-\frac{\gamma}{2}(N')^{\frac12})$$

for any $|n| \leq N'$, *and any* $N \leq N' \leq N''$ *In particular, the limits*

$$E(x,\omega) := \lim_{N'\to\infty} E_{j_{N'}}^{(N')}(x,\omega)$$
$$\psi(x,\omega,n) := \lim_{N'\to\infty} \psi_{j_{N'}}^{(N')}(x,\omega,n), \quad n \in \mathbb{Z}$$

exist,

$$|E(x,\omega) - E_j^{(N)}(x,\omega)| \leq 2\exp(-\frac{\gamma}{2}N^{\frac12}), \quad |\psi(x,\omega,n) - \psi_{j_N}^{(N)}(x,\omega,n)| \leq \exp(-\frac{\gamma}{2}(N)^{\frac12}) \tag{13.2}$$

and $\sum_n |\psi(x,\omega,n)|^2 = 1$. *Furthermore,*

$$H(x,\omega)\psi(x,\omega,\cdot) = E(x,\omega)\psi(x,\omega,\cdot) \tag{13.3}$$

$$\lim_{n\to\infty}\frac{1}{2n}\log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) = -L(\omega,E(x,\omega)) \tag{13.4}$$

(6) *Let* $\omega \in \mathbb{T}_{c,a} \setminus \hat{\Omega}_N$, $x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$, *and* $E^{(N)}_{j_m}(x,\omega) \in (E', E'')$, $m = 1, 2$, $j_1 \neq j_2$. *Then*

$$|E^{(N)}_{j_1}(x,\omega) - E^{(N)}_{j_2}(x,\omega)| \geq \exp(-N^\delta)$$

Let $E_m(x,\omega)$ *be the eigenvalue of* $H(x,\omega)$, *defined in (5) above, which obeys*

$$|E_m(x,\omega) - E^{(N)}_{j_m}(x,\omega)| < 2\exp(-\frac{\gamma}{2} N^{\frac{1}{2}})$$

for $m = 1, 2$. *Then*

$$|E_1(x,\omega) - E_2(x,\omega)| > \frac{1}{2}\exp(-N^\delta).$$

If $(E', E'') = (-\infty, +\infty)$, *then for each* $|j| \leq N/2$ *one has an "almost Parseval identity"*

$$0 \leq 1 - \sum_m |\langle \delta_j(\cdot), \psi_m(x,\omega,\cdot)\rangle|^2 \leq \exp(-\frac{\gamma}{8} N)$$

where $\delta_j(\cdot)$ *stands for the* δ*–function at* $n = j$. *The collection of all eigenfunctions* $\psi_m(x,\omega,\cdot)$ *obtained this way for* $N, N+1, \ldots$ *form a complete orthonormal system in* ℓ^2.

(7) *Let* $\omega \in \mathbb{T}_{c,a} \setminus \hat{\Omega}_N$, $x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$. *Let* $\{\psi_m(x,\omega,\cdot)\}$ *be all possible eigenfunctions of* $H(x,\omega)$ *defined in (5) above for all scales* $\geq N$. *Let* $J_{x,\omega}$ *be be the closure of the set of the corresponding eigenvalues. If* $E \in (E', E'') \setminus J_{x,\omega}$, *then* $(H(x,\omega) - E)$ *is invertible. In other words,*

$$\operatorname{spec}\big(H(x,\omega)\big) \cap (E', E'') = J_{x,\omega},$$

and the functions $\psi_m(x,\omega,\cdot)$ *form a complete orthonormal system in the spectral subspace of* $H(x,\omega)$ *corresponding to* (E', E'').

(8) *Assume that* $\nu^{(N)}_j(x,\omega) \in [-N + N^{1/2}, N - N^{1/2}]$. *Then for any* k *such that*

$$-N + N^{1/2}/2 < \nu^{(N)}_j(x,\omega) + k < N - N^{1/2}/2$$

there exists a unique

$$E^{(N)}_{j_k}(x + k\omega, \omega) \in \operatorname{spec}\big(H_{[-N,N]}(x + k\omega, \omega)\big)$$

such that

$$\big|E^{(N)}_j(x,\omega) - E^{(N)}_{j_k}(x + k\omega,\omega)\big| < \exp\big(-\gamma N^{1/2}/4\big)\ , \tag{13.5}$$

$$E^{(N)}_{j_k}(x + k\omega,\omega) \notin \mathcal{E}_{N,\omega}\ , \tag{13.6}$$

$$\big|\nu^{(N)}_{j_k}(x + k\omega,\omega) - \big(\nu^{(N)}_j(x,\omega) + k\big)\big| \leq N^{1/2}/4\ , \tag{13.7}$$

$$-N + N^{1/2}/4 < \nu^{(N)}_{j_k}(x + k\omega,\omega) < N - N^{1/2}/4\ , \tag{13.8}$$

$$\sum_{|m + k - \nu^{(N)}_j(x,\omega)| \leq N^{1/2}/4} \big|\psi^{(N)}_{j_k}(x + k\omega, m) - \psi^{(N)}_j(x, m + k)\big|^2 < \exp\big(-\gamma N^{1/2}/8\big) \tag{13.9}$$

(9) *Let* $\omega \in \mathbb{T}_{c,a} \setminus \hat{\Omega}_N$, $x_0 \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$. *Then for any* $x \in \mathbb{T}$ *one has*

$$\operatorname{spec}\big(H(x,\omega)\big) \cap (E', E'') = J_{x_0,\omega}$$

In other words, $\operatorname{spec}\big(H(x,\omega)\big) \cap (E', E'')$ *is the same for all* $x \in \mathbb{T}$

(10) *For any* $x \in \mathbb{T}$ *and any* N *one has*

$$\big[(\operatorname{spec}(H(x,\omega)) \setminus \mathcal{S}_{N,\omega}) \cup (\mathcal{S}_{N,\omega} \setminus \operatorname{spec}(H(x,\omega)))\big] \cap (E', E'') \subset \bigcup_{N' \geq N} \mathcal{E}_{N',\omega}$$

(11) *If* $\mathrm{spec}(H(x,\omega)) \cap (E', E'') \neq \emptyset$ *for some* x *then*

$$\mathrm{mes}\,(\mathrm{spec}(H(x,\omega))) \cap (E', E'') > 0$$

If $(E', E'') = (-\infty, \infty)$ *then*

$$\mathrm{mes}\,(\mathrm{spec}(H(x,\omega))) \geq \exp(-(\log N_0)^{C_1})$$

Most of the preparatory work needed for the proof of this proposition has already been done in the previous sections. We need only few more auxiliary statements.

Lemma 13.2. *There exists* $N_0 = N_0(V, c, a, \gamma)$ *such that for any* $N \geq N_0$, $\omega \in \mathbb{T}_{c,a}$, $E \in (\underline{E}, \overline{E})$ *the following property holds: if* $\mathrm{dist}(E, \mathcal{S}_{N,\omega}) \geq \exp(-N^{\frac{1}{2}})$, *then* $\mathrm{dist}(E, \mathcal{S}_{\overline{N},\omega}) \geq \frac{1}{2}\exp(-N^{\frac{1}{2}})$ *for any* $\overline{N} \geq N$.

Proof. Due to Corollary 2.20, $\mathrm{dist}\left[E, \mathrm{spec}(H_N(x,\omega))\right] \geq \frac{1}{2}\exp(-N^{\frac{1}{2}})$ implies

$$\log|f_{[1,N]}(e(x), \omega, E)| > \gamma N - N^{\frac{3}{4}} \tag{13.10}$$

provided $N \geq N_0(V, c, a, \gamma)$. Due to Lemma 6.1, (13.10) implies

$$|(H_{[1,N]}(x,\omega) - E)^{-1}(m,n)| \leq \exp(-\frac{\gamma}{4}N)$$

for any $m, n \in [1, N]$, $|m - n| \geq \frac{N}{2}$. Therefore, $\mathrm{dist}(E, \mathcal{S}_{N,\omega}) \geq \frac{1}{2}\exp(-N^{\frac{1}{2}})$ implies

$$|(H_{[N'+1,N'+N]}(x,\omega) - E)^{-1}(m,n)| \leq \exp(-\frac{\gamma}{4}N) \tag{13.11}$$

for any $x \in \mathbb{T}$, $N' \in \mathbb{Z}$, $m, n \in [N'+1, N'+N]$, $|m-n| \geq \frac{N}{2}$. Let $N \leq \overline{N} \leq \exp(N^{\frac{3}{4}})$. Assume that

$$H_{[1,\overline{N}]}(x,\omega)\psi(x,n) = E\psi(x,n) \tag{13.12}$$

Let $\mu := \max_{1 \leq n \leq \overline{N}} |\psi(x,n)|$. Then, due to (13.11) and the "Poisson formula" one obtains

$$|\psi(x,n)| \leq 2\mu\exp(-\frac{\gamma}{4}N), \text{ for any } n \in [1, \overline{N}]$$

Hence,

$$\mu \leq 2\mu\exp(-\frac{\gamma}{4}N)$$

This yields $\mu = 0$, and thus $\psi(x,n) = 0$ for any $n \in [1, \overline{N}]$. Thus

$$\mathrm{dist}(E, \mathcal{S}_{N,\omega}) \geq \frac{1}{2}\exp(-N^{\frac{1}{2}}) \Longrightarrow E \notin \mathrm{spec}\left(H_{[1,\overline{N}]}(x,\omega)\right) \quad \forall\, x \in \mathbb{T} \tag{13.13}$$

The lemma follows from (13.13). ☐

Lemma 13.3. *There exists* $N_0 = N_0(V, c, a, \gamma, A)$ *such that for any* $N \geq N_0$, $\omega \in \mathbb{T}_{c,a}$, $E \in (\underline{E}, \overline{E})$ *the following property holds: if* $\mathrm{dist}\left[E, \mathrm{spec}\left(H_{[1,N]}(x,\omega)\right)\right] \geq \exp(-(\log N)^A)$, *then*

$$|(H_{[1,N]}(x,\omega) - E)^{-1}(m,n)| \leq \exp(-L(\omega,E)|m-n| + (\log N)^{2A}) \tag{13.14}$$

for any $m, n \in [1, N]$.

Proof. Due to Corollary 2.20, $\mathrm{dist}\left[E, \mathrm{spec}\left(H_N(x,\omega)\right)\right] \geq \exp(-(\log N)^A)$ implies

$$\log|f_{[1,N]}(e(x), \omega, E)| > L(\omega, E)N - (\log N)^{A+C} \tag{13.15}$$

provided $N \geq N_0(V, c, a, \gamma, A)$. As in the proof of Lemma 6.1 one sees that (13.15) implies (13.14). ☐

Lemma 13.4. *Given $\varepsilon > 0$ sufficiently small, there exists $N_0 = N_0(V, c, a, \gamma, \varepsilon)$ such that for any $N \ge N_0$, $\omega \in \mathbb{T}_{c,a}$, $x \in \mathbb{T}$, $E \in (\underline{E}, \overline{E})$ the following assertion holds. Assume that*

$$H(x,\omega)\psi(\cdot) = E\psi(\cdot) \tag{13.16}$$

for some E and some function $\psi(n)$, $n \in \mathbb{Z}$. Assume also that

$$\max_{1\le n\le N} |\psi(n)| = 1 \tag{13.17}$$

Then

$$\min_{1\le n\le N} \frac{1}{2}\log(|\psi(n)|^2 + |\psi(n-1)|^2) \ge -N(L(\omega,E)+\varepsilon) \tag{13.18}$$

Proof. Recall that for any $a < b$

$$\begin{bmatrix}\psi(b+1)\\ \psi(b)\end{bmatrix} = M_{[a,b]}(e(x),\omega,E)\begin{bmatrix}\psi(a)\\ \psi(a-1)\end{bmatrix} \tag{13.19}$$

where $M_{[a,b]}(e(x),\omega,E)$. Recall also that due to the uniform upper estimate of Lemma 2.10 for any $E \in (\underline{E}, \overline{E})$

$$\sup_{0<b-a\le N, x\in\mathbb{T}} \|M_{[a,b]}(e(x),\omega,E)\| \le \exp(N(L(\omega,E)+\varepsilon))$$

provided $N \ge N_0 = N_0(V,c,a,\gamma,\varepsilon)$. Since

$$\|M^{-1}\| = \|M\|$$

for any unimodular matrix M, one obtains

$$\exp(-N(L(\omega,E)+\varepsilon))\Big\|\begin{bmatrix}\psi(a)\\ \psi(a-1)\end{bmatrix}\Big\| \le \Big\|\begin{bmatrix}\psi(b+1)\\ \psi(b)\end{bmatrix}\Big\| \le \exp(N(L(\omega,E)+\varepsilon))\Big\|\begin{bmatrix}\psi(a)\\ \psi(a-1)\end{bmatrix}\Big\| \tag{13.20}$$

for any $0 < b-a \le N$. Due to the assumptions of the lemma there exist $a \in [1,N]$ such that

$$1 \le \Big\|\begin{bmatrix}\psi(a)\\ \psi(a-1)\end{bmatrix}\Big\| \le 2$$

Therefore the statement follows from (13.20). □

We now turn to the proof of Proposition 13.1.

Proof of Proposition 13.1. (1) Let $\Omega_N, \mathcal{E}_{N,\omega}$ be the subset defined in Lemma 6.2. Let $\omega \in \mathbb{T}_{c,a}$. Set

$$\mathcal{B}_{N,\omega} = \{x \in \mathbb{T} : \operatorname{spec}(H_{[-N,N]}(x,\omega)) \cap (E', E'') \cap \mathcal{E}_{N,\omega} \ne \emptyset\}$$

The subsets $\Omega_N, \mathcal{E}_{N,\omega}, \mathcal{B}_{N,\omega}$ satisfy properties (1), (2) of Proposition 13.1.

(3) Clearly, $\mathcal{S}_{N,\omega}$ is a union of $(2N+1)$ closed intervals. Set

$$\mathcal{E}'_{N,\omega} = \{E \in \mathbb{R} : E \notin \mathcal{S}_{N,\omega}, \operatorname{dist}(E, \mathcal{S}_{N,\omega}) \le \exp(-N^{\frac{1}{2}})\}$$

Note that $\operatorname{mes}(\mathcal{E}'_{N,\omega}) \lesssim N\exp(-N^{\frac{1}{2}})$, $\operatorname{compl}(\mathcal{E}'_{N,\omega}) \lesssim N$. With some abuse of notation we denote the set $\mathcal{E}_{N,\omega} \cup \mathcal{E}_{N',\omega}$ as $\mathcal{E}_{N,\omega}$. Clearly $\mathcal{E}_{N,\omega}$ obeys property (1) of Proposition 13.1. Due to Proposition 13.2, property (3) of Proposition 13.1 holds.

(4) Let $E_j^{(N)}(x,\omega) \in (E', E'')$ for some $x \in \mathbb{T}\setminus\mathcal{B}_{N,\omega}$. Then due to Lemma 6.2 there exists $\nu_j^{(N)}(x,\omega) \in [-N,N]$ such that

$$|\psi_j^{(N)}(x,\omega,n)| \le \exp\big(-\frac{\gamma}{2}|n-\nu_j^{(N)}(x,\omega)|\big)$$

provided $|n-\nu_j^{(N)}(x,\omega)| > N^{\frac{1}{2}}$. So, part (4) of Proposition 13.1 is valid.

(5) Let $\omega \in \mathbb{T}_{c,a}\backslash\hat{\Omega}_N$, $x \in \mathbb{T}\backslash\, \hat{\mathcal{B}}_{N,\omega}$. Assume $\nu_j^{(N)}(x,\omega) \in [-N+N^{\frac{1}{2}}, N-N^{\frac{1}{2}}]$. Then due to standard perturbation theory of Hermitian matrices, for each $N \le N' \le N$ there exists an eigenvalue $E_j^{(N')}(x,\omega)$ of $H_{[-N',N']}(x,\omega)$ such that the estimates (13.1) hold. Assume that $x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$. Then (13.1) applies for any $N' \ge N$. Therefore the limits

$$E(x,\omega) = \lim_{N'\to\infty} E_{j_{N'}}^{(N')}(x,\omega)$$
$$\psi(x,\omega,n) = \lim_{N'\to\infty} \psi_{j_{N'}}^{(N')}(x,\omega,n), \quad n \in \mathbb{Z}$$

exist, relations (13.2), (13.3) hold and

$$\sum_n |\psi(x,\omega,n)|^2 = 1 \tag{13.21}$$

Pick $N_1 \in \mathbb{Z}$ so that $N \asymp (\log N_1)^B$. Since $\omega \in \mathbb{T}_{c,a}\backslash\hat{\Omega}_N$, $x \in \mathbb{T}\backslash\, \hat{\mathcal{B}}_{N,\omega}$ due to Proposition 9.9 (applied with $t = 2N$) one has with A=4B

$$\operatorname{dist}\big[E_{j_{N_1}}^{(N_1)}(x,\omega), \operatorname{spec}\big(H_{[3N,N_1]}(x,\omega)\big)\big] \ge \exp(-(\log N_1)^A),$$
$$\operatorname{dist}\big[E_{j_{N_1}}^{(N_1)}(x,\omega), \operatorname{spec}\big(H_{[-N_1,-3N]}(x,\omega)\big)\big] \ge \exp(-(\log N_1)^A)$$

That implies

$$\begin{aligned}\operatorname{dist}\big[E(x,\omega), \operatorname{spec}\big(H_{[3N,N_1]}(x,\omega)\big)\big] &\ge \exp(-(\log N_1)^A),\\ \operatorname{dist}\big[E(x,\omega), \operatorname{spec}\big(H_{[-N_1,-3N]}(x,\omega)\big)\big] &\ge \exp(-(\log N_1)^A)\end{aligned} \tag{13.22}$$

Due to Lemma 13.3 relation (13.22) in its turn implies the following estimates

$$\begin{aligned}|(H_{[3N,N_1]}(x,\omega) - E(x,\omega))^{-1}(m,n)| &\le \exp(-L(\omega,E(x,\omega))|m-n| + (\log N_1)^{2A})\\ |(H_{[-N_1,-3N]}(x,\omega) - E(x,\omega))^{-1}(m,n)| &\le \exp(-L(\omega,E(x,\omega))|m-n| + (\log N_1)^{2A})\end{aligned} \tag{13.23}$$

for any m,n in the corresponding interval. Let n be arbitrary such that

$$(\log N_1)^{3A} \le |n| \le N_1/2$$

Applying Poisson's formula to $\psi(x,\omega,n)$ one obtains

$$\begin{aligned}|\psi(x,\omega,n)| &\le \exp\big[-L(\omega,E(x,\omega))(|n| - 3N - (\log N_1)^{2A})\big]\\ &\quad + \exp\big[-L(\omega,E(x,\omega))(N_1 - |n| - (\log N_1)^{2A})\big]\\ &\lesssim \exp\big[-L(\omega,E(x,\omega))|n|(1-o(1))\big]\end{aligned}$$

as $n \to \infty$. This estimate implies

$$\limsup_{n\to\infty} \frac{1}{2n} \log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) \le -L(\omega,E(x,\omega)) \tag{13.24}$$

Note that using the above notations one has

$$\max_{|n|\le N} |\psi(x,\omega,n)| \ge 1/N$$

since $|\psi(x,\omega,n)|$ is normalized. Thus, in view of Lemma 13.4,

$$\min_{-N\le n\le N_1} \frac{1}{2} \log(|\psi(x,\omega,n)|^2 + |\psi(n-1)|^2) \ge -N(L(\omega,E)+\varepsilon)$$

provided $N_1 \ge N_0(V,c,a,\gamma,\varepsilon)$. Hence

$$\liminf_{n\to\infty} \frac{1}{2n} \log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) \ge -L(\omega,E(x,\omega)) - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary

$$\liminf_{n\to\infty} \frac{1}{2n} \log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) \ge -L(\omega,E(x,\omega))$$

Thus

$$\lim_{n\to\infty} \frac{1}{2n} \log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) = -L(\omega, E(x,\omega))$$

Similarly

$$\lim_{n\to-\infty} \frac{1}{2n} \log(|\psi(x,\omega,n)|^2 + |\psi(x,\omega,n+1)|^2) = -L(\omega, E(x,\omega))$$

as claimed.

(6) Let $\omega \in \mathbb{T}_{c,a} \backslash \hat{\Omega}_N$, $x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$. Assume that $E_{j_m}^{(N)}(x,\omega) \in (E', E'')$, and

$$\nu_{j_m}^{(N)}(x,\omega) \in [-N + N^{\frac{1}{2}}, N - N^{\frac{1}{2}}], \quad m = 1, 2$$

Then due to Proposition 7.1 one has

$$\big|E_{j_1}^{(N)}(x,\omega) - E_{j_2}^{(N)}(x,\omega)\big| > \exp(-N^\delta) \tag{13.25}$$

Let $E_m(x,\omega), \psi_m(x,\omega,\cdot)$ be defined as in (5) for $j = j_m$, $m = 1, 2$. Then, from (13.2) and (13.25),

$$\big|E_1(x,\omega) - E_2(x,\omega)\big| > \frac{1}{2}\exp(-N^\delta) \tag{13.26}$$

If $(E', E'') = (-\infty, \infty)$, then part (5) is applicable to each eigenvalue $E_j^{(N)}(x,\omega)$, provided

$$\nu_j^{(N)}(x,\omega) \in [-N + N^{\frac{1}{2}}, N - N^{\frac{1}{2}}]. \tag{13.27}$$

Furthermore, let $\psi_{j_m}^{(N)}(x,\omega,\cdot)$ be all eigenfunctions of $H_{[-N,N]}(x,\omega)$ with

$$-N + N^{\frac{1}{2}} < \nu_{j_m}^{(N)}(x,\omega) < N - N^{\frac{1}{2}}$$

Let $\psi_m(x,\omega,\cdot)$ be the eigenfunction defined in part (5) for $j = j_m$ where $1 \le j \le m_0 \le 2N+1$. Let $\delta_k(\cdot)$ be the delta-function at $n = k$, $k \in [-N + 4N^{1/2}, N - 4N^{1/2}]$. Clearly, one has

$$|\langle \delta_k, \psi_j^{(N)}(x,\omega,\cdot)\rangle| \le \exp(-\gamma N^{1/2}/2)$$

for any j with $\nu_j^{(N)}(x,\omega) \notin [-N+N^{1/2}, N-N^{1/2}]$. Since $\{\psi_j^{(N)}(x,\omega,\cdot)\}_{j=1}^{2N+1}$, form an orthonormal basis in the space of all functions on $[-N,N]$, one has

$$0 \le 1 - \sum_{m=1}^{m_0} |\langle \delta_k(\cdot), \psi_m(x,\omega,\cdot)\rangle|^2 \le \exp(-\frac{\gamma}{2}N^{\frac{1}{2}})$$

(7) Let $J_{x,\omega}$ be the closure of the set of all eigenvalues $E_m(x,\omega)$ of $H(x,\omega)$ defined in (5). Let $E_0 \in (E', E'')$. Assume $\sigma_0 := \mathrm{dist}(E_0, J_{x,\omega}) > 0$. Then due to the definition of the eigenvalues $E_m(x,\omega)$ one has for any $N \ge N_0$

$$\min\Big\{|E_0 - E_j^{(N)}(x,\omega)| : E_j^{(N)}(x,\omega) \in (E', E''), \nu_j^{(N)}(x,\omega) \in [-N + N^{\frac{1}{2}}, N - N^{\frac{1}{2}}]\Big\} \ge \sigma_0/2$$

Let $\varphi(n)$, $n \in \mathbb{Z}$ be an arbitrary normalized $\ell^2(\mathbb{Z})$ function supported on some interval $[-T, T]$. Let $N > 2T$. Then $|\langle \varphi, \psi_j^{(N)}(x,\omega,\cdot)\rangle| \le \exp(-N^{2/3})$ for any j with $\nu_j^{(N)}(x,\omega) \notin [-N+N^{1/2}, N-N^{1/2}]$. Hence,

$$\sum_{\nu_j^{(N)}(x,\omega) \in [-N+N^{1/2}, N-N^{1/2}]} |\langle \varphi, \psi_j^{(N)}(x,\omega,\cdot)\rangle|^2 \ge 1/2$$

Expanding $\varphi(\cdot)$ in the basis $\{\psi_j^{(N)}(x,\omega,\cdot)\}_{j=1}^{2N+1}$ one obtains

$$\|(H_{[-N,N]}(x,\omega) - E_0)\varphi\|^2 \ge (1/2)(\sigma_0/2)^2$$

Hence

$$\|(H(x,\omega) - E_0)\varphi\|^2 \ge (1/2)(\sigma_0/2)^2$$

Since φ here is arbitrary one infers that

$$\mathrm{dist}(H(x,\omega), E_0) \gtrsim \sigma_0$$

Thus part (7) holds.

(8) Part (8) is due to Lemma 11.4.

(9) Given x, let $\mathcal{M}(x,N,s)$ be the collection of all m such that the eigenfunction $\psi^{(N)}_{j_m}(x,\omega,\cdot)$ defined as in (5) obeys

$$-N + sN^{\frac12} < \nu_j^{(N)}(x,\omega) < N - sN^{\frac12}, \quad s = 1,2,3$$

Let $J_{x,\omega,s}$ be the closure of all eigenvalues $E_m(x,\omega)$ defined in (5) for all N and $m \in \mathcal{M}(x,N,s)$. Then just as above one has $\mathrm{spec}(H(x,\omega)) \cap (E', E'') = J_{x,\omega,s}$. On the other hand, fix $x_0 \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$, and take arbitrary k such that $-N/2 < \nu_j^{(N)}(x,\omega) + k < N/2$. Let $x = x_0 + k\omega \pmod 1$. Then $x_0, x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{2N,\omega}$. Due to part (8) and part (5) for each $m \in \mathcal{M}(x_0, 2N, 3)$, there exists $m' \in \mathcal{M}(x, 2N, 2)$ such that

$$E^{(2N)}_{j_m}(x_0,\omega) - E^{(2N)}_{j_{m'}}(x,\omega) \le \exp(-N^{1/3})$$

Hence $J_{x_0,\omega,3} \subset \{E \ : \ \mathrm{dist}(E, J_{x,\omega,2}) \le \exp(-N^{1/3})\}$. Thus

$$J_{x_0,\omega} = J_{x_0,\omega,3} \subset J_{x,\omega,2} = J_{x,\omega}$$

Switching the roles of x_0 and x in this argument implies $J_{x_0,\omega} = J_{x,\omega}$ provided

$$x_0 \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}, \quad x = x_0 + k\omega \pmod 1$$

for some $|k| \le N/2$. Since $\hat{\mathcal{B}}_{N',\omega} \subset \hat{\mathcal{B}}_{N,\omega}$ for $N' \ge N$ the claim is valid for any $x = x_0 + k\omega \pmod 1$ with arbitrary k. Given arbitrary x' and $\rho > 0$, one can find k such that with $x = x_0 + k\omega \pmod 1$ one has $|x' - x| < \rho$. Then

$$\mathrm{spec}(H(x',\omega)) \subset \{E : \mathrm{dist}\big(E, \mathrm{spec}(H(x,\omega))\big) \le \rho\}$$

and

$$\mathrm{spec}(H(x,\omega)) \subset \{E : \mathrm{dist}\big(E, \mathrm{spec}\, H(x',\omega)\big) \le \rho\}$$

Since $\mathrm{spec}(H(x,\omega)) = J_{x_0,\omega}$, and ρ is arbitrary, one has $\mathrm{spec}(H(x',\omega)) = J_{x_0,\omega}$. Thus (9) is valid.

(10) Part (10) follows from (7) combined with part (3).

(11) If $\mathrm{spec}(H(x,\omega)) \cap (E', E'') \ne \emptyset$ for some x, then $J_{x_0,\omega} \cap (E',E'') \ne \emptyset$ for some $x_0 \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N,\omega}$ with some N. Therefore, using the notations from the proof of part (9) one has

$$J_{x_0,\omega,1} \cap (E', E'') \ne \emptyset$$

Hence $E^{(2N)}_{j_m}(x_0,\omega) \in (E', E'')$. It follows from Proposition 10.11 that $\mathcal{S}_{N,\omega} \cap (E', E'')$ contains an I-segment, $|I| \ge \exp(-(\log N)^C)$. Due to part (10) this implies

$$\mathrm{mes}\,(\mathrm{spec}(H(x,\omega))) \cap (E', E'') \ge \exp(-(\log N)^C)/2$$

which proves the first claim of (11). The second claim (11) is implicit in the argument leading to the first part and we are done.

□

Before we proceed with the proof of Theorem 1.3 we make the following remark which we will use in the proof of Theorem 1.1. It follows from inspection of the proof of assertion (7) of Proposition 13.1.

Remark 13.5. *Let $\omega \in \mathbb{T}_{c,a} \backslash \hat{\Omega}_{N_1}$, $x_0 \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N_1,\omega}$, $E_0 \in \mathbb{R}$, and let $\sigma_0 > 0$ be a constant. Assume that for any $N_2 \ge N_1$ there exists $N \ge N_2$ such that*

$$\min\Big\{|E_0 - E_j^{(N)}(x_0,\omega)| \ : \ E_j^{(N)}(x_0,\omega) \in (E', E''), \nu_j^{(N)}(x_0,\omega) \in [-N + N^{\frac12}, N - N^{\frac12}]\Big\} \ge \sigma_0 \tag{13.28}$$

Then for any $x \in \mathbb{T}$ one has $E_0 \notin \mathrm{spec}(H(x,\omega))$.

To proceed we need the following general statement.

Lemma 13.6. *Let $\omega \in \mathbb{T}_{c,a}$. Given $x \in \mathbb{T}$ and $N \in \mathbb{Z}$, $N > 0$ set*

$$\mu(x,\omega,N) := \min_{t\in\mathbb{Z},|t|\le N} \|x - t\omega\|$$

Let $0 < \alpha < \beta < 1$ be arbitrary. There exists $N_0 = N_0(c,a,\alpha,\beta)$ such that the following statement holds: Let $x \in \mathbb{T}$, and $x \notin \omega\mathbb{Z} \pmod 1$. Assume that $\|x - t_1\omega\| \le \exp(-|t_1|^\beta)$ for some $t_1 \in \mathbb{Z}$, $|t_1| \ge N_0$. Then there exists $s > t_1$ such that for any $s/4 \le s_1 \le s$ holds

$$\exp(-s^\beta) \lesssim \mu(x,\omega,s_1) \lesssim \exp(-s^\alpha)$$

Proof. Given arbitrary N_0 set ($\mathcal{R}$ here stands for "recurrence")

$$\mathcal{R}(x,\omega,N_0) := \{t \in \mathbb{Z} : |t| \ge N_0, \|x - t\omega\| \le \exp(-|t|^\beta)\}$$

If $t' \in \mathcal{R}(x,\omega,N_0)$ and $t \in \mathbb{Z}$, $|t| \le |t'|$, $t \ne t'$, then $\|x - t\omega\| > \|x - t'\omega\|$ provided N_0 is large enough. Indeed, otherwise

$$\|(t - t')\omega\| \le 2\exp(-|t'|^\beta)$$

and $|t' - t| \le 2|t'|$ which contradicts the condition $\omega \in \mathbb{T}_{c,a}$. Hence for each $t' \in \mathcal{R}(x,\omega,N_0)$

$$\mu(x,\omega,|t'|) = \|x - t'\omega\|$$

Assume first that $\mathcal{R}(x,\omega,N_0)$ consists of a single integer t_1. Let $T := |t_1|$. Note that $\mu(x,\omega,T) > 0$ since $x \notin \omega\mathbb{Z} \pmod 1$. Set $s := [(\log \mu(x,\omega,T)^{-1})^{\frac{1}{\alpha}}]$. Then

$$s \ge [|t_1|^{\frac{\beta}{\alpha}}] > 4|t_1| = 4T$$

because of $\alpha < \beta$. Let $t \in \mathbb{Z}$ be such that $T < |t| \le s$. Since t_1 is the only element of $\mathcal{R}(x,\omega,N)$ one has $t \notin \mathcal{R}(x,\omega,N)$. Hence,

$$\|x - t\omega\| > \exp(-|t|^\beta) \ge \exp(-s^\beta)$$

If $|t| \le T$, then due to the definition of $\mu(x,\omega,T)$ one has

$$\|x - t\omega\| \ge \mu(x,\omega,T) \ge \exp(-(s+1)^\alpha) \asymp \exp(-s^\alpha)$$

Since $\mu(x,\omega,T) = \|x - t_1\omega\| \asymp \exp(-s^\alpha)$ one concludes that

$$\exp(-s^\beta) \lesssim \mu(x,\omega,s_1) \lesssim \exp(-s^\alpha)$$

for any $T < s_1 \le s$. Since $T < s/4$, the lemma follows in this case. Assume now that there exist at least two points in $\mathcal{R}(x,\omega,N)$, viz. $t_i \in \mathcal{R}(x,\omega,N_0)$, $i = 1,2$, $t_1 \ne t_2$. We can assume that $|t_1| < |t_2|$ and also that

$$|t_2| = \min\{|t| : t \in \mathcal{R}(x,\omega,N_0) \setminus \{t_1\}, |t| \ge |t_1|\}$$

Then

$$\|(t_1 - t_2)\omega\| \le 2\max_i \|x - t_i\omega\| = 2\|x - t_1\omega\|$$

since

$$\|x - t_1\omega\| = \mu(x,\omega,|t_1|) \ge \mu(x,\omega,|t_2|) = \|x - t_2\omega\|$$

Since $\omega \in \mathbb{T}_{c,a}$, that implies in particular

$$|t_2 - t_1| \ge \|x - t_1\omega\|^{-1/3} \ge \exp(-\frac{1}{3}|t_1|^\beta)$$

provided N_0 is large enough. Hence

$$|t_2| \ge \frac{1}{2}\|x - t_1\omega\|^{-1/3} \ge \frac{1}{2}\mu(x,\omega,|t_1|)^{-1/3}$$

Define T and s just as before. Then $4|t_1| < s < |t_2|$. Moreover, if $|t_1| < |t| \le s$, then $t \notin \mathcal{R}(x,\omega,N_0)$ due to our choice of t_2. Therefore, one can just repeat the argument from the case $\mathcal{R}(x,\omega,N_0) = \{t_1\}$. $\square$

Proof of Theorem 1.3. We will follow the notations of Proposition 13.1. Set

$$\Omega := \bigcap_{N \ge N_0} \Omega_N, \quad \mathcal{B}_\omega := \bigcap_{N \ge N_0} \bigcup_{|k| \le N} (\tilde{\mathcal{B}}_{N,\omega} + k\omega) \pmod 1$$

where

$$\tilde{\mathcal{B}}_{N,\omega} := \{x \in \mathbb{T} : \mathrm{dist}(x, \hat{\mathcal{B}}_{N,\omega}) \le \exp(-N^{1/5})\}$$

These sets are decreasing and it follows from part (1) of Proposition 13.1 that they are of Hausdorff dimension zero. Moreover, The set $\mathcal{B}_\omega$ is invariant under the shifts $x \mapsto x + m\omega$, $m \in \mathbb{Z}$. Let

$$\omega \in (\mathbb{T}_{c,a} \cap (\omega_0 - \rho^{(0)}, \omega_0 + \rho^{(0)})) \setminus \Omega,$$

Due to part (11) of Proposition 13.1 $\mathrm{mes}\,(\Sigma_\omega) > 0$. Due to part (5) of Proposition 13.1 for any any $x \in \mathbb{T} \backslash \mathcal{B}_\omega$ eigenfunctions $\psi_j(x, \omega, \cdot)$, $j = 1, \ldots$ are defined,

$$\lim_{|N| \to \infty} \frac{1}{2N} \log(|\psi_j(x, \omega, N)|^2 + |\psi_j(x, \omega, N+1)|^2) = -L(\omega, E_j(x, \omega))$$

and functions $\psi_j(x, \omega, \cdot)$ form an orthonormal basis in $\ell^2(\mathbb{Z})$. Moreover, the eigenvalues $E_j(x, \omega)$ are simple. Let $E_0 \in \mathbb{R}$ be arbitrary. Assume that there exist $x(k) \in \mathbb{T} \setminus \mathcal{B}_\omega$, $1 \le k \le k_0 \in \mathbb{Z}^+ \cup \{\infty\}$ such that

(i) for each k there exists $j(k)$ such that $E_0 = E_{j(k)}(x(k))$
(ii) the orbits $\Gamma(x(k))$ are all different, i.e., for any $k_1 \ne k_2$ and any $t \in \mathbb{Z}$ one has $x(k_2) \ne x(k_1) + t\omega \pmod 1$

To finish the proof of the theorem we have to evaluate k_0. There exists N_0 such that $x(k) \in \mathbb{T} \setminus \tilde{\mathcal{B}}_{N,\omega}$ for each $1 \le k \le k_0$ and all $N \ge N_0$. It follows from parts (5) and (6) of Proposition 13.1 that for each k there exists j_k such that $-N + N^{\frac12} < \nu_{j_k}^{(N)}(x, \omega) < N - N^{\frac12}$ and

$$|E_0 - E_{j_k}^{(N)}(x(k), \omega)| \le \exp(-N^{1/3})$$

Fix arbitrary k. Due to part (8) of Proposition 13.1 for each $s \in \mathbb{Z}$ with

$$-N + N^{\frac12} < \nu_{j_k}^{(N)}(x, \omega) + s < N - N^{\frac12}$$

(we call these s admissible in this proof) there exists $j_{k,s}$ such that

$$|E_0 - E_{j_{k,s}}^{(N)}(x(k) + s\omega, \omega)| \lesssim \exp(-N^{1/3}) \tag{13.29}$$

Set $z_{k,s} := e(x(k) + s\omega)$. Due to (13.29) and Corollary 2.20 there exists $\zeta_{k,s} \in \mathcal{D}(z_{k,s}, \exp(-N^{1/3}))$ such that $f_N(\zeta_{k,s}, \omega, E_0) = 0$. Recall the following estimate (see Remark 4.5)

$$\mathcal{M}_N(\omega, E, 1/2, 2) := (2N+1)^{-1} \#\big\{z \,:\, z \in \mathcal{A}_{\rho_0},\ f_N(z, \omega, E) = 0\big\} \le C(V) < \infty \tag{13.30}$$

Assume that $k_0 > C(V)$. Then there exist $k_1 \ne k_2$ and admissible s_1,s_2 such that $\zeta_{k_1,s_1} = \zeta_{k_2,s_2}$, $|z_{k_1,s_1} - z_{k_2,s_2}| \lesssim \exp(-N^{1/3})$. Hence

$$\|x(k_1) - x(k_2) - s(k_1,k_2)\omega\| \lesssim \exp(-N^{1/3}) \tag{13.31}$$

with some $s(k_1,k_2) \in \mathbb{Z}$, $|s(k_1,k_2)| \le 2N$. Due to Lemma 13.6 either there exists N_1, depending on $(x(k_1) - x(k_2))$ such that for any $t \ge N_1$

$$\|x(k_1) - x(k_2) - t\omega\| \gtrsim \exp(-t^{1/3})$$

or there exists $N \ge N_1$, $t \in \mathbb{Z}$, $|t| \le N/4$ such that

$$\exp(-N^{1/3}) \lesssim \|x(k_1) - x(k_2) - t\omega\| \lesssim \exp(-N^{1/4})$$

Recall that $N' \le N''$ implies that $\tilde{\mathcal{B}}_{N'',\omega} \subset \tilde{\mathcal{B}}_{N',\omega}$. Therefore, due to the argument which leads to (13.31), one can assume that the latter case takes place. Hence, we can replace relation (13.31) by the following one

$$\exp(-N^{1/3}) \lesssim \|x(k_1) - x(k_2) - s(k_1,k_2)\omega\| \lesssim \exp(-N^{1/4}) \tag{13.32}$$

where $|s(k_1,k_2)| \le N/4$. Since $x(k_i) \in \mathbb{T}\setminus\tilde{\mathcal{B}}_{N,\omega}$, due to part (5) of Proposition 13.1 there exists a segment $\{E_{j_i}^{(N)}(\cdot,\omega)\ ,\underline{x}_i,\overline{x}_i\}$ such that $x(k_i) \in (\underline{x}_i + \exp(-N^{1/5}),\ \overline{x}_i - \exp(-N^{1/5}))$ and

$$|E_{j(k_i)}(x(k_i),\omega) - E_{j_i}^{(N)}(x(k_i),\omega)| \le \exp(-\gamma N^{\frac{1}{2}})$$

$i = 1, 2$. Note that due to part (5) of Proposition 13.1 we can also assume that

$$|\nu_{j_i}^{(N)}(x(k_i),\omega)| \le N/4 \tag{13.33}$$

$i = 1, 2$ since otherwise we can just replace N by $N' = 2N$. Set

$$\hat{x}(k_1) := x(k_2) + s(k_1,k_2)\omega \pmod 1$$

Then $\hat{x}(k_1) \in (\underline{x}_1,\ \overline{x}_1)$. Since $\exp(-N^{1/3}) \lesssim |x(k_1) - \hat{x}(k_1)|$ one has

$$|E_{j_1}^{(N)}(x(k_1),\omega) - E_{j_1}^{(N)}(\hat{x}(k_1),\omega)| \gtrsim \exp(-2N^{1/3})$$

On the other hand due to part (8) of Proposition 13.1 and since $|s(k_1,k_2)| \le N/4$

$$|E_{j_1}^{(N)}(\hat{x}(k_1),\omega) - E_{j_2}^{(N)}(x(k_2),\omega)| \lesssim \exp(-N^{1/2})$$

Since $E_{j(k_1)}(x(k_1),\omega) = E_{j(k_2)}(x(k_2),\omega)$ we arrive at a contradiction. Thus $k_0 \le C(V)$. If $V(e(x))$ is a trigonometric polynomial then $C(V) \le 2\deg(V(e(\cdot)))$ due to Remark 4.5. $\square$

14. Elimination of triple resonances

The goal of this section is to eliminate ω with the property that for some $x \in \mathbb{T}$ and some $0 < m < m'$

$$\begin{aligned} |E_{j_1}^{(n)}(x+m\omega,\omega) - E_{j_0}^{(n)}(x,\omega)| &< \varepsilon \\ |E_{j_2}^{(n)}(x+m'\omega,\omega) - E_{j_0}^{(n)}(x,\omega)| &< \varepsilon \end{aligned} \tag{14.1}$$

with three distinct j_0, j_1, j_2. Of course it will be important to specify the mutual sizes of n, m, m' and ε. Unlike the previous elimination machinery based on the resultant of two polynomials from Section 5, the elimination here will be based on the implicit function theorem; this in turn will require the lower bound on the slopes of the $E_j^{(N)}(\cdot,\omega)$ that was obtained above at the expense of eliminating a small set of energies. Note that there is no hope of eliminating the situation described by (14.1) unless the $E_j^{(N)}(x,\omega)$ truly depend on x (for example, if the potential is constant – this is of course excluded in our case since we are assuming positive Lyapunov exponents). We have chosen to present the elimination process "abstractly" at first, i.e., without any reference to the $E_j^{(N)}$. Later we will apply the abstract elimination theorem, see Proposition 14.7, to the system (14.1). We now begin with a number of standard calculus lemmas that develop the implicit function theorem in a quantitative way. We use the well-known idea of basing the implicit function theorem on monotonicity arguments (this is of course restricted to *scalar* implicit functions). The first lemma is nothing but a careful statement of a quantitative implicit function theorem.

Lemma 14.1. (1) *Let $f \in C^1(\mathcal{R})$ where $\mathcal{R} := (a,b)\times(c,d)$. Assume that*

$$\mu := \inf_{(x,y)\in\mathcal{R}} \partial_y f(x,y) > 0, \quad K := \sup_{(x,y)\in\mathcal{R}} |\partial_x f(x,y)| < \infty \tag{14.2}$$

If $f(x_0,y_0) = 0$ for some $(x_0,y_0) \in \mathcal{R}$, then for any

$$x \in J_0 := (x_0 - \kappa_0, x_0 + \kappa_0) \cap (a,b), \qquad \kappa_0 := h_0\mu K^{-1}, \quad h_0 := \min(y_0 - c, d - y_0)$$

there exists a unique $y = \phi_0(x) \in (c,d)$ such that $f(x,\phi_0(x)) = 0$. Moreover, $\phi_0 \in C^1(J_0)$ and $\sup_{x\in J_0} |\phi_0'(x)| \le K\mu^{-1}$.

(2) *Assume in addition that the function $f(x, y)$ admits an analytic continuation to the domain*

$$\mathcal{T} = \{(z, w) \in \mathbb{C}^2 : \mathrm{dist}\big((z, w), \mathcal{R}\big) < r_0\} \tag{14.3}$$

for some $r_0 > 0$, and obeys

$$\sup_{\mathcal{T}} |f(z, w)| \le 1 \tag{14.4}$$

Then the implicit function ϕ_0 has an analytic continuation to the rectangle $U = J_0 \times (-r, r)$ where

$$r := \min(r_0^3 \mu^2, r_0)/8 \tag{14.5}$$

Furthermore, $f(z, \phi_0(z)) = 0$ for all $z \in U$ and $\sup_{z \in U} |\phi_0(z)| \le \max(|c|, |d|) + r_0$.

Proof. Note that for any $x \in (a, b)$ one has $|f(x, y_0)| \le K|x - x_0|$. In particular, for any $|x - x_0| < \kappa_0$

$$|f(x, y_0)| < h_0 \mu$$

Given such x consider the case $0 < f(x, y_0) < h_0\mu$. Since $c \le y_0 - h_0 < y_0 + h_0 < d$, we infer that

$$f(x, y_0 - h_0) < h_0\mu - h_0\mu = 0.$$

Hence, there exists a unique $y = \phi_0(x) \in (y_0 - h_0, y_0)$ such that $f(x, \phi_0(x)) = 0$. If instead $-h_0\mu < f(x, y_0) \le 0$ then there exists a unique $y = \phi_0(x) \in (y_0, y_0 + h_0)$ such that $f(x, \phi_0(x)) = 0$. It follows from the chain rule that $\phi_0 \in C^1(J_0)$ and $|\phi_0'(x)| \le K\mu^{-1}$; in fact,

$$\phi_0'(x) = -\frac{\partial_x f(x, \phi_0(x))}{\partial_y f(x, \phi_0(x))}$$

for all $x \in J_0$. That finishes the proof of part (1). To prove part (2) fix an arbitrary $x_1 \in J_0$ and set $y_1 = \phi_0(x_1)$. Due to Cauchy's estimates one has

$$|\partial_w^n f(x_1, y_1 + w)|_{w=0} \le n! r_0^{-n} \qquad \forall\, n \ge 0$$

Since $f(x_1, y_1) = 0$, $\partial_w f(x_1, y_1 + w)|_{w=0} \ge \mu$ the Taylor series expansion for $f(x_1, y_1 + w)$ in the disk $\mathcal{D}(0, r_1)$, with $r_1 := \min(r_0^2\mu/2, r_0)$ yields

$$|f(x_1, y_1 + w)| \ge \mu r_1/2$$

for any $|w| = r_1$. Furthermore, $w \mapsto f(x_1, y_1 + w)$ has a simple zero at $w = 0$ in the disk $|w| \le r_1$. Applying Cauchy's estimate in the z-variable now implies that

$$|f(x_1 + z, y_1 + w)| \ge r_0^2\mu^2/8, \quad \forall\, |z| \le r,\ |w| = r_1$$

where r is as in the statement of the lemma. We now claim that

$$\frac{1}{2\pi i} \oint_{|w|=r_1} \frac{f_w(x_1 + z, y_1 + w)}{f(x_1 + z, y_1 + w)}\, dw = 1$$

for all $|z| < r$. Indeed, the integral on the left counts the number of zeros $f(x_1 + z, y_1 + \cdot) = 0$ inside the disk $|w| < r_1$ and with z fixed. Since there is a unique zero at $w = 0$ in this disk when $z = 0$ is fixed, and since the integral is analytic in $|z| < r$ the claim follows. By the residue theorem, the sought after implicit function is given by

$$\frac{1}{2\pi i} \oint_{|w|=r_1} w \frac{f_w(z, y_1 + w)}{f(z, y_1 + w)}\, dw = \phi_0(z)$$

for all $|z - x_1| < r$. It is analytic and has all the desired properties. Covering all of J_0 with such disks we obtain ϕ_0 on U by the uniqueness of analytic continuations. □

The next lemma is a slight variant which does not require vanishing at a point but only smallness. We of course reduce the latter case to the former.

Lemma 14.2. *Let $f \in C^1(\mathcal{R})$ be as in the previous lemma and suppose μ and K are as in* (14.2). *Assume that $|f(x_1, y_1)| < \varepsilon$ for some $(x_1, y_1) \in \mathcal{R}$ and $0 < \varepsilon \le h_1\mu$ where $h_1 := \min(y_1 - c, d - y_1)/2$. Then*

(1) *for any* $x \in J_1 := (x_1 - \kappa_1, x_1 + \kappa_1) \cap (a, b)$ *with* $\kappa_1 := h_1\mu K^{-1}$ *there exists a unique* $y = \phi_1(x) \in (c, d)$ *such that* $f(x, \phi_1(x)) = 0$. *Moreover,* $\phi_1 \in C^1(J_1)$ *and*

$$\sup_{x\in J_1} |\phi_1'(x)| \leq K\mu^{-1}$$

(2) *for any* $x \in J_1$ *and any* $y \in (c, d) \setminus (\phi_1(x) - \varepsilon\mu^{-1}, \phi_1(x) + \varepsilon\mu^{-1})$ *one has* $|f(x, y)| \geq \varepsilon$

Assume in addition that f *admits an analytic continuation to the domain* (14.3) *and obeys condition* (14.4). *Then* ϕ_1 *admits an analytic continuation to the rectangle* $V := J_1 \times (-r, r)$ *with* r *as in* (14.5). *One has* $\sup_{z\in V} |\phi_1(z)| \leq \max(|c|, |d|) + r_0$ *and* $f(z, \phi_1(z)) = 0$ *for all* $z \in V$.

Proof. Assume for instance that $0 \leq f(x_1, y_1) < \varepsilon$. Since $c < y_1 - h_1 < d$, we conclude that $f(x_1, y_1 - h_1) < \varepsilon - \mu h_1 \leq 0$. Hence, there exists a unique $\tilde{y}_1 \in (y_1 - h_1, y_1]$ such that $f(x_1, \tilde{y}_1) = 0$. By Lemma 14.1 with $(x_0, y_0) := (x_1, \tilde{y}_1)$ there exists a C^1-function $\phi_1(x)$ defined on the interval

$$J_2 := (x_1 - \kappa_0, x_1 + \kappa_0) \cap (a, b)$$

with

$$\kappa_0 := h_0\mu K^{-1}, \quad h_0 = \min(\tilde{y}_1 - c, d - \tilde{y}_1)$$

such that $f(x, \phi_1(x)) = 0$ for any $x \in J_2$. Moreover, $\sup_{x\in J_1} |\phi_1'(x)| \leq K\mu^{-1}$. Note first that

$$h_0 \geq \min(y_1 - c, d - y_1) - h_1 = h_1$$

by construction. So, $\phi_1(x)$ is defined on the interval J_1. Clearly, $|f(x, y)| \geq \varepsilon$ for any $x \in J_1$ and any $y \in (c, d) \setminus (\phi_1(x) - \varepsilon\mu^{-1}, \phi_1(x) + \varepsilon\mu^{-1})$. That proves the first part of the statement. The statement about the analytic continuation of ϕ_1 follows from part (2) of Lemma 14.1. □

We can now combine these local lemmas with a covering procedure to obtain a global result. It is a quantitative version of the following qualitative statement: suppose $f = f(x, y)$ is smooth on some rectangle $\mathcal{R}$ and $\partial_y f \neq 0$ on $\mathcal{R}$. Then the set where $|f| < \varepsilon$ in $\mathcal{R}$ with $\varepsilon > 0$ small is covered by a union of neighborhoods of the (local) graphs $(x, \phi(x))$ where $f(x, \phi(x)) = 0$. Figure 3 shows a possible form of the set $U_f(h, \varepsilon)$ appearing in the following proposition (indicated by the shaded areas). The big rectangle is $\mathcal{R}$ and the two horizontal dashed lines are at heights $c + h_1$ and $d - h_1$, respectively, defining a smaller rectangle. Note that due to the fact that $f(x, y)$ is increasing in y, the dashed areas cannot have any x-projection in common. Also, they cannot "die" inside of the smaller rectangle due to the previous lemmas (implicit function theorem). Hence, they can only end on the boundaries of the smaller rectangle. Also note that while each of the two shaded areas is defined by graphs over x, they are not graphs over y. However, we can cut them up into finitely many graphs over y, see below.

Proposition 14.3. *Let* $f \in C^1(\mathcal{R})$ *and* μ, K *be as in* (14.2). *Given* $0 < h_1 < (d-c)/4$, *and* $0 < \varepsilon \leq h_1\mu$, *there is a sequence of pairwise disjoint intervals* $\{J_i\}_{i=1}^m$ *in* (a, b), *with (provided they are arranged in increasing order)*

$$\min_{2\leq i\leq m-1} |J_i| \geq \kappa_1 := h_1\mu K^{-1}$$

and satisfying the following properties:

(1) *for each* $1 \leq i \leq m$ *there exists* $\phi_i \in C^1(J_i)$ *such that*

$$f(x, \phi_i(x)) = 0 \quad \forall\, x \in J_i, \quad \sup_{x\in J_i} |\phi_i'(x)| \leq K\mu^{-1}$$

(2) *the set*

$$\mathcal{U}_f(h_1, \varepsilon) := \{(x, y) \in (a, b) \times (c + h_1, d - h_1) \;:\; |f(x, y)| < \varepsilon\}$$

satisfies

$$\mathcal{U}_f(h_1, \varepsilon) \subset \bigcup_{i=1}^m \mathcal{S}_x(\phi_i, \varepsilon\mu^{-1})$$

where for any $\sigma > 0$,

$$\mathcal{S}_x(\phi_i, \sigma) := \{(x, y) \;:\; x \in J_i,\; y \in (\phi_i(x) - \sigma, \phi_i(x) + \sigma)\}$$

Furthermore, assume in addition that f admits an analytic continuation to the domain (14.3) *and obeys condition* (14.4)*. Then for each i the function ϕ_i admits an analytic continuation to the domain*

$$\mathcal{T}_i = \{z = x + iy \;:\; x \in J_i,\ |y| < r\}$$

with r as in (14.5)*, and $f(z, \phi_i(z)) = 0$ as well as $|\phi_i(z)| \le \max(|c|, |d|) + r_0$ for all $z \in \mathcal{T}_i$.*

Proof. By Lemma 14.2, for each $(x_1, y_1) \in \mathcal{U}_f(h_1, \varepsilon)$ there exists an interval $J(x_1, y_1) := J_1$ and a function $y = \phi_1(x)$ as described in that lemma. This defines a collection

$$\mathcal{C} := \{J(x_1, y_1)\}_{(x_1,y_1)\in\mathcal{U}_f(h_1,\varepsilon)}$$

Suppose $J(x_1, y_1), J(\tilde{x}_1, \tilde{y}_1) \in \mathcal{C}$ have a nonempty intersection and let ϕ_1 and $\tilde{\phi}_1$ denote the functions associated with $J(x_1, y_1)$ and $J(\tilde{x}_1, \tilde{y}_1)$, respectively. Then

$$\phi_1(x) = \tilde{\phi}_1(x) \quad \forall\, x \in J(x_1, y_1) \cap J(\tilde{x}_1, \tilde{y}_1)$$

due to the monotonicity of $y \mapsto f(x, y)$. Define an equivalence relation on the intervals in $\mathcal{C}$ as follows: $J, \tilde{J} \in \mathcal{C}$ are equivalent iff they can be connected by a chain of pairwise intersecting intervals in $\mathcal{C}$. Then we find the J_i in the statement above simply by taking the union over all intervals in an equivalence class. The ϕ_i are well-defined by the aforementioned uniqueness property of the graphs. The analytic continuation statement follows from Lemma 14.2. $\square$

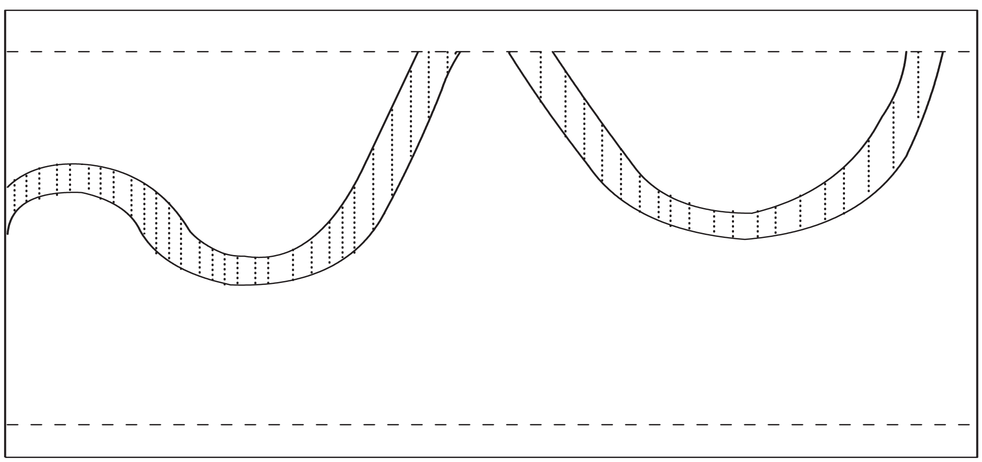

FIGURE 3. The set $\mathcal{U}_f(h, \varepsilon)$

In view of (14.1) we will need to apply the previous proposition to a system $|f(x, y)| < \varepsilon$, $|g(x, y)| < \varepsilon$. More precisely, we wish to eliminate a small set of y so that this system fails for every x assuming suitable lower bounds on $\partial_y f(x, y)$ and $\partial_y g(x, y)$ (one of these derivatives will need to be much larger than the other). We shall proceed by eliminating x from the system $|f(x, y)| < \varepsilon$, $|g(x, y)| < \varepsilon$. Note that this cannot be done on the basis of Proposition 14.3 alone as we will need to invert each function $y = \phi_i(x)$. This will of course not always be possible. However, by a Sard type argument we will be able to remove a small set of y (in measure) so that the inversion can be carried out for the remaining y.

The precise formulation of this procedure is given by Proposition 14.6 below. We start with the following lemma, which is a quantitative version of Sard's theorem.

Lemma 14.4. *Let ϕ be a real–valued function defined on the interval (α, β) which admits an analytic continuation to the domain*

$$\mathcal{S} = \{z \in \mathbb{C} \;:\; \operatorname{dist}(z, (\alpha, \beta)) < r\}$$

for some $r > 0$ and satisfies $\sup_{\mathcal{S}} |\phi(z)| \le q$ for some $q > 0$. Then, given $0 < \delta < \frac{q}{5r}$ there exist at most

$$n \le 12r^{-1}(\beta - \alpha + r)\log(qr^{-1}\delta^{-1})$$

disjoint intervals $I_j \subset (\alpha, \beta)$ such that

$$|\phi'(x)| \ge \delta, \qquad \forall x \in \bigcup_j I_j \tag{14.6}$$

$$|\phi'(x)| \le 2\delta, \qquad \forall x \in (\alpha,\beta)\backslash \bigcup_j I_j \tag{14.7}$$

In particular,

$$\int_{(\alpha,\beta)\backslash \bigcup_j I_j} |\phi'(x)|\, dx \le 2\delta(\beta - \alpha) \tag{14.8}$$

Proof. Let $x_1 \in (\alpha, \beta)$ be arbitrary and define $(\alpha_1, \beta_1) := (x_1 - \frac{r}{8}, x_1 + \frac{r}{8})$. By a standard covering argument, it suffices to consider (α_1, β_1) in the role of (α, β). The function ϕ' is analytic on $\mathcal{D}(x_1, r)$ and due to Cauchy's estimates satisfies

$$\max_{\mathcal{D}(x_1, 7r/8)} |\phi'(z)| \le 8qr^{-1}$$

Assume first that there exists $\zeta_1 \in \mathcal{D}(x_1, r/8)$ such that

$$|\phi'(\zeta_1)| \ge 2\delta \tag{14.9}$$

Then, due to the standard Jensen formula (4.1) applied to $\phi'(z) \pm \delta$ in the disk $\mathcal{D}(\zeta_1, 3r/4)$ one obtains

$$\begin{aligned}\#\{z \in \mathcal{D}(\zeta_1, r/4) \,:\, \phi'(z) \pm \delta = 0\} &\le \int_0^1 \log|\phi'(\zeta_1 + \frac{3}{4} re(\theta)) \pm \delta|\, d\theta - \log|\phi'(\zeta_1)| \\ &\le \log(8qr^{-1} + \delta) - \log(2\delta) \le \log(5qr^{-1}\delta^{-1}) \\ &\le 2\log(qr^{-1}\delta^{-1})\end{aligned}$$

Hence there exist at most

$$n_1 \le 2\log(qr^{-1}\delta^{-1}) + 1 \le 3\log(qr^{-1}\delta^{-1})$$

disjoint intervals $I_j \subset (\alpha_1, \beta_1)$ such that

$$\begin{aligned}&|\phi'(x)| \ge \delta, \;\; \forall x \in \bigcup_j I_j \\ &|\phi'(x)| \le \delta, \;\; \forall x \in (\alpha_1, \beta_1)\backslash \bigcup_j I_j\end{aligned}$$

If condition (14.9) fails then

$$|\phi'(x)| \le 2\delta, \;\; \forall x \in (\alpha_1, \beta_1)$$

which finishes the proof. □

Remark 14.5. *In Lemma 14.4 we set up δ as a "dimensionless" quantity in the following sense: consider the scaling $\phi_\lambda(x) := \lambda^{-1}\phi(\lambda x)$ for any $\lambda > 0$. Clearly, ϕ'_λ scales like λ^0 which is what we mean by (14.6) and thus δ, being scaling invariant. Note, however, that q scales like λ^{-1} which explains why the $q\delta^{-1}r^{-1}$ term inside the logarithm above is scaling invariant (which of course is necessary). The somewhat strange scaling comes from the context of the implicit function theorem. Indeed, let $f_\lambda(x, y) := f(\lambda x, \lambda y)$ which simply means that we homothetically scale the rectangle $\mathcal{R}$ to $\lambda^{-1}\mathcal{R}$. Then the implicit function $y = \phi(x)$ scales precisely as $\phi_\lambda(x)$ above. It is instructive to check our "abstract" results in this section against this scaling. For example, in Lemma 14.1 both K and μ scale like λ, whereas ϕ'_0 scales like λ^0 as required by the estimate $|\phi'_0(x)| \le K\mu^{-1}$. We will keep all "abstract" results in this section scaling invariant.*

We are now able to formulate the aforementioned "$x = \psi(y)$" version of Proposition 14.3. In Figure 3 this corresponds to removing those pieces from the two shaded areas where the graphs defining the boundaries have horizontal tangents.

Proposition 14.6. *Let $f(x,y)$ be a real-valued function defined in $\mathcal{R} = (a,b) \times (c,d)$. Assume that f admits an analytic continuation to the domain (14.3) and obeys condition (14.4). Let μ, K be as in (14.2). Apply Proposition 14.3 to f with parameters h_1, ε as specified there, and let ϕ_i be the resulting functions defined on the intervals J_i, $1 \le i \le m$. Fix an arbitrary*

$$0 < \delta < \frac{q}{5r}, \qquad r = \min(r_0^3\mu^2, r_0)/8, \quad q := \max(|c|,|d|) + r_0 \tag{14.10}$$

see (14.5). Then for each $1 \le i \le m$ there exist pairwise disjoint intervals

$$J_{i,j} \subset J_i, \quad \forall\, 1 \le j \le j(i),$$

so that for each of these intervals

$$\phi_i \,:\, J_{i,j} \to I_{i,j} := \phi_i(J_{i,j})$$

is invertible. Denote the inverse by $\psi_{i,j}$. Then the following properties hold:

1. *The total number of intervals $J_{i,j}$ (and thus of $I_{i,j}$ as well as of $J'_{i,k}$ which are defined to be the connected components of $J_i \setminus \bigcup_{j=1}^{j(i)} J_{i,j}$) is at most*
$$M := 12\big((b-a)(r^{-1} + \kappa_1^{-1}) + 2\big)\, \log(qr^{-1}\delta^{-1})$$
where $\kappa_1 := h_1 \mu K^{-1}$.
2. *there exist no more than $3M$ many intervals I_ℓ such that $\sum_\ell |I_\ell| \le 2(b-a)\delta + 3M\varepsilon\mu^{-1}$ and so that with $\mathcal{U}_f(h_1,\varepsilon)$ as in Proposition 14.3,*
$$\mathcal{U}_f(h_1,\varepsilon) \setminus \Big(\bigcup_{i=1}^{m}\bigcup_{j=1}^{j(i)} J_{i,j} \times I_{i,j}\Big) \subset (a,b) \times \bigcup_\ell I_\ell \tag{14.11}$$
3. *for any $\sigma > 0$, define*
$$\mathcal{S}_y(\psi_{i,j},\sigma) := \{(x,y) \,:\, y \in I_{i,j},\ |\psi_{i,j}(y) - x| < \sigma\}$$
Then
$$\mathcal{U}_f(h_1,\varepsilon) \bigcap \Big(\bigcup_{i=1}^{m}\bigcup_{j=1}^{j(i)} J_{i,j} \times I_{i,j}\Big) \subset \bigcup_{i=1}^{m}\bigcup_{j=1}^{j(i)} \mathcal{S}_y(\psi_{i,j}, \varepsilon\delta^{-1}\mu^{-1}) \tag{14.12}$$
4. *$|\psi'_{i,j}(y)| \le \delta^{-1}$ for all $y \in I_{i,j}$ and all i,j*

Proof. By Proposition 14.3 for each i the function ϕ_i admits an analytic continuation in the domain

$$\mathcal{T}_i = \{z = x + iy : x \in J_i, |y| \le r\}$$

with $\sup_{\mathcal{T}_1} |\phi_0(z)| \le q$. Applying Lemma 14.4 to each ϕ_i produces pairwise disjoint intervals

$$J_{i,j} \subset J_i, \quad 1 \le j \le j(i) \le n_i$$

with, cf. (14.10),

$$n_i \le 12(|J_i|r^{-1} + 1)\log(qr^{-1}\delta^{-1})$$

such that

$$|\phi'_i(x)| \ge \delta \quad \forall\, x \in \bigcup_{j=1}^{j(i)} J_{i,j}, \tag{14.13}$$

$$\sum_i \int_{J_i \setminus \bigcup_{j=1}^{j(i)} J_{i,j}} |\phi'_i(x)|\, dx \le \sum_i 2|J_i|\delta \le 2(b-a)\delta \tag{14.14}$$

Since $J_i \ge \kappa_1$ for all but possibly two i, one has

$$\sum_{i=1}^{m} n_i \le 12\big((b-a)(r^{-1} + \kappa_1^{-1}) + 2\big)\, \log(qr^{-1}\delta^{-1})$$

as claimed in property (1). To prove property (2) of the proposition, we define I_ℓ the be the collection of all intervals arising as follows: $\varepsilon\mu^{-1}$-neighborhoods of all $\phi_i(J'_{i,k})$ (we refer to these as type I), as well as $\varepsilon\mu^{-1}$-neighborhoods of each of the two points in $\phi_i(\partial J_{i,j})$ (these are type II). By property (1) there are no more than M type I intervals, as well as at most $2M$ type II intervals. Moreover, each type II interval has measure not exceeding $2\varepsilon\mu^{-1}$, whereas all type I intervals in total have measure at most

$$\sum_{i,k} \big(|\phi_i(J'_{i,k})| + 2\varepsilon\mu^{-1}\big) \le 2(b-a)\delta + 2M\varepsilon\mu^{-1},$$

see (14.14). To prove (14.11), observe from Proposition 14.3 that it suffices to prove the inclusion

$$\bigcup_{i=1}^{m} \Big[\mathcal{S}_x(\phi_i, \varepsilon\mu^{-1}) \setminus \bigcup_{j=1}^{j(i)} J_{i,j} \times I_{i,j}\Big] \subset (a,b) \times \Big[(c,d) \setminus \bigcup_{\ell} I_\ell\Big]$$

A point (x, y) belongs to the set inside the brackets on the left-hand side iff either of the following two scenarios occurs

- $x \in J_i \setminus \bigcup_{j=1}^{j(i)} J_{i,j} = \bigcup_k J'_{i,k}$ and $|y - \phi_i(x)| < \varepsilon\mu^{-1}$
- $x \in J_{i,j}$ and $|y - \phi_i(x)| < \varepsilon\mu^{-1}$ but $y \notin I_{i,j} = \phi_i(J_{i,j})$

In the first case, $y \in I_\ell$ where I_ℓ is a type-I interval, whereas in the second case, $y \in I_\ell$ which is type-II. In conclusion,

$$\mathcal{S}_x(\phi_i, \varepsilon\mu^{-1}) \setminus \bigcup_{j=1}^{j(i)} J_{i,j} \times I_{i,j} \subset J_i \times \Big[(c,d) \setminus \bigcup_{\ell} I_\ell\Big]$$

which yields the desired property by taking unions over the pairwise disjoint J_i. On each interval $I_{i,j} := \phi_i(J_{i,j})$ an inverse function $\psi_{i,j} = \phi_i^{-1}$ is defined, and moreover

$$\sup_{y \in I_{i,j}} |\psi'_{i,j}(y)| \le \delta^{-1}$$

from (14.13). This establishes property (4). Note that if $y \in I_{i,j}$ and $|y - \phi_i(x)| < \varepsilon\mu^{-1}$ for some $x \in J_{i,j}$, then $|x - \psi_{i,j}(y)| \le \varepsilon\delta^{-1}\mu^{-1}$. In other words, in view of Proposition 14.3 one has

$$\mathcal{U}_f(h_1, \varepsilon) \cap \Big(J_{i,j} \times I_{i,j}\Big) \subset \mathcal{S}_y(\psi_{i,j}, \varepsilon\mu^{-1}\delta^{-1})$$

which implies property (3) above. □

We are now in a position to state and prove the main "abstract" elimination result of this section. It will be applied to (14.1).

Proposition 14.7. *Let $f(x, y)$ be a real-valued function defined in $\mathcal{R} = (a, b) \times (c, d)$. Assume that f admits an analytic continuation to the domain*

$$\mathcal{T} = \{(z, w) \in \mathbb{C}^2 : \operatorname{dist}\big((z, w), \mathcal{R}\big) < r_0\}$$

for some $r_0 > 0$, and obeys $\sup_{\mathcal{T}} |f(z, w)| \le 1$. Furthermore, let $g \in C^1(\widetilde{\mathcal{R}})$ be a real-valued function defined on $\widetilde{\mathcal{R}} := (a, b) \times (c', d')$. Assume that

$$\mu := \inf_{(x,y)\in\mathcal{R}} \partial_y f(x, y) > 0, \quad \bar\mu := \inf_{(x,y)\in\widetilde{\mathcal{R}}} \partial_y g(x, y) > 0$$

$$K := \sup_{(x,y)\in\mathcal{R}} |\partial_x f(x, y)| + \sup_{(x,y)\in\widetilde{\mathcal{R}}} |\partial_x g(x, y)| < \infty$$

Set $r := \min(r_0^3\mu^2, r_0)/8$, $q := \max(|c|, |d|) + r_0$ and pick $h_1 \in (0, (d - c)/4)$, $\sigma_1 \in (0, \frac{q}{5r})$, $\sigma_2 \in (0, h_1\mu)$. Define $\kappa_1 := h_1\mu K^{-1}$, as well as

$$M := 12\big((b - a)(r^{-1} + \kappa_1^{-1}) + 2\big) \log(qr^{-1}\sigma_1^{-1}) \tag{14.15}$$

and assume that

$$\bar\mu \ge \max\big[4(1 + K\sigma_1^{-1}\mu^{-1})M, 2K\sigma_1^{-1}\big]$$

Then there exist subintervals $\{U_\ell\}_{\ell=1}^{\ell_0}$ *of* (c,d) *and* $\{V_k\}_{k=1}^{k_0}$ *of* (c',d') *so that*

(i) $k_0 \le \ell_0 \le 3M$
(ii) $\sum_{\ell=1}^{\ell_0} |U_\ell| \le 2\sigma_1(b-a) + 3M\sigma_2\mu^{-1}$, $\sum_{k=1}^{k_0} |V_k| \le 2\sigma_2(b-a)$
(iii) *the intervals* U_ℓ *only depend on the function* f*, the rectangle* $\mathcal{R}$*, and* σ_1, σ_2
(iv) *for any*

$$y \in (c+h_1, d-h_1) \cap (c',d') \setminus \bigcup_{\ell,k} U_\ell \cup V_k$$

and any $x \in (a,b)$ *at least one of the following two inequalities fails:*

$$|f(x,y)| < \sigma_2, \quad |g(x,y)| < \sigma_2 \tag{14.16}$$

Proof. We apply Proposition 14.6 to f with $\delta = \sigma_1, \varepsilon := \sigma_2$. This produces intervals $J_{i,j}$ and $I_{i,j}$ as well as functions $\psi_{i,j}$ defined on $I_{i,j}$ satisfying properties (1)–(4) in that proposition. First, we define $\{U_\ell\}_{\ell=1}^{\ell_0}$ to be the same as the I_ℓ, see property (2) of the proposition. Hence, properties (i)–(iii) from above pertaining to $\{U_\ell\}$ follow immediately from Proposition 14.6. To define the V_k, observe the following: by the chain rule, for any $y \in I_{i,j} \cap (c',d')$

$$\frac{d}{dy} g(\psi_{i,j}(y), y) \ge \partial_y g(\psi_{i,j}(y), y) - |\psi_{i,j}'(y)||\partial_x g(\psi_{i,j}(y), y)| \ge \bar{\mu} - \sigma_1^{-1}K \ge \frac{\bar{\mu}}{2}$$

since we are assuming that $\bar{\mu} \ge 2K\sigma_1^{-1}$. Hence, given $\beta > 0$ there exists $I_{i,j}' \subset I_{i,j} \cap (c',d')$ such that

$$|I_{i,j}'| \le 4\beta\bar{\mu}^{-1} \tag{14.17}$$

and

$$|g(\psi_{i,j}(y), y)| > \beta \qquad \forall\, y \in I_{i,j} \cap (c',d') \setminus I_{i,j}'$$

Note that we allow for the possibility that $I_{i,j} \cap (c',d') \setminus I_{i,j}' = \emptyset$ or $I_{i,j}' = \emptyset$. Now define the intervals $\{V_k\}_{k=1}^{k_0}$ to be the entire collection $\{I_{i,j}' \cap (c',d')\}_{i,j}$ (skipping empty $I_{i,j}'$), with the choice of $\beta := \sigma_2(1 + K\sigma_1^{-1}\mu^{-1})$. Note that $k_0 \le j_0$ and property (i) is done. Moreover, by (1) of the previous proposition,

$$\sum_k |V_k| = \sum_{i,j} |I_{i,j}'| \le 4\beta\bar{\mu}^{-1}M \le \sigma_2$$

by our lower bound on $\bar{\mu}$. This finishes property (ii) above. To prove property (iv), let

$$(x,y) \in (a,b) \times \Big[(c+h_1, d-h_1) \cap (c',d') \setminus \bigcup_{\ell,k} U_\ell \cup V_k\Big] \tag{14.18}$$

We can of course assume that $(x,y) \in \mathcal{U}_f(h_1, \sigma_2)$ for otherwise $|f(x,y)| \ge \sigma_2$. Because of (14.18) and (14.11), we then have

$$(x,y) \in \mathcal{U}_f(h_1,\sigma_2) \bigcap \bigcup_{i=1}^{m} \bigcup_{j=1}^{j(i)} J_{i,j} \times (I_{i,j} \setminus I_{i,j}') \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{j(i)} \mathcal{S}_y(\psi_{i,j}, \sigma_2\sigma_1^{-1}\mu^{-1}),$$

where the inclusion is (14.12). Now note the following: fix i,j so that $(x,y) \in J_{i,j} \times I_{i,j}$. Since $|f(x,y)| < \sigma_2$,

$$|x - \psi_{i,j}(y)| \le \sigma_2\sigma_1^{-1}\mu^{-1}$$

Hence,

$$|g(x,y)| \ge |g(\psi_{i,j}(y), y)| - K|x - \psi_{i,j}(y)| > \beta - K\sigma_2\sigma_1^{-1}\mu^{-1} > \sigma_2$$

provided we choose $\beta = \sigma_2(1 + K\sigma_1^{-1}\mu^{-1})$ as before. In the final step we used that

$$(x,y) \in J_{i,j} \times (I_{i,j} \cap (c',d') \setminus I_{i,j}').$$

But this means that $|g(x,y)| \ge \sigma_2$ assuming (14.18) and $|f(x,y)| < \sigma_2$, and the proposition is proved. $\square$

Next, we apply this result to establish the main elimination result concerning system (14.1). We will use the notion of an segment from Definition 11.2. Let $\left\{E_j^{(m)}(\cdot,\omega), \underline{x}', \bar{x}'\right\}$ be a segment. In the following corollary we will work with the following quantitative properties of segments:

(a) $|\partial_x E_j^{(m)}(\cdot,\omega)| \geq e^{-m^{\delta_1}}$, for any $x \in (\underline{x}', \bar{x}')$

(b) $\bar{x}' - \underline{x}' \geq e^{-m^{\delta_0}}$

where $0 < \delta_0,\ \delta_1 \ll 1$ are some parameters. Note that these are weaker than the ones implicit in Definition 11.2, cf. (11.1). Furthermore, due to Remark 11.3, given x, ω the function $E_j^{(m)}(\cdot,\cdot)$ admits an analytic continuation to the polydisk

$$\mathcal{P}(x,\omega,m) := \{(z,w) \in \mathbb{C}^2 : |z - e(x)| < r(m), |w-\omega| < r(m)\}$$

where $r(m) := \exp(-m^{\delta_0})$. Moreover,

$$\sup_{\mathcal{P}(x,\omega,m)} |E_j^{(m)}(z,w)| \leq C(V)$$

Recall the sets Ω_m, $\mathcal{E}_{m,\omega}$, and $\widetilde{\mathcal{E}}_{m,\omega}$ from Section 11 (the latter obviously depends on parameters δ_0, δ_1). In the following corollary,

$$\widetilde{\Omega}_n := \bigcup_{n \leq m \leq 100n} \Omega_m$$

Corollary 14.8. *Choose $A \geq 10$, $d \geq 4$ arbitrary*[9] *but fixed, as well as $0 < \varepsilon_0 < \frac{1}{20}$ and $0 < 2\delta_0 \leq \delta_1 \leq \frac{\varepsilon_0}{A}$. Let $N \geq N_0(V, \rho_0, a, c, \gamma, \delta_0, \delta_1, A, d, \varepsilon_0)$ be large and set $n := [(\log N)^A]$. Then there exist $\mathcal{B}_n', \mathcal{B}_n'' \subset \mathbb{T}$ so that*

$$\begin{aligned} &\operatorname{mes}(\mathcal{B}_n') < N^{-\varepsilon_0}, \quad \operatorname{compl}(\mathcal{B}_n') \leq \exp((\log N)^{1/2}) \\ &\operatorname{mes}(\mathcal{B}_n'') < N^{-d+3}, \quad \operatorname{compl}(\mathcal{B}_n'') \leq N^3 \end{aligned}$$

with the following property: for all

$$\omega \in \mathbb{T}_{c,a} \setminus (\mathcal{B}_n' \cup \mathcal{B}_n'' \cup \widetilde{\Omega}_n)$$

and for each choice of $n \leq n_1, n_2, n_3 \leq 100n$ and $1 \leq j_i \leq 2n_i + 1$, $i = 1,2,3$, as well as every

$$e^{n^{3\delta_1}} \leq m_1 \leq \exp((\log N)^{1/4}), \quad N^{10\varepsilon_0} \leq m_2 \leq 2N$$

the system

$$\begin{aligned} &|E_{j_2}^{(n_2)}(x + m_1\omega, \omega) - E_{j_1}^{(n_1)}(x,\omega)| < N^{-d} \\ &|E_{j_3}^{(n_3)}(x + m_2\omega, \omega) - E_{j_1}^{(n_1)}(x,\omega)| < N^{-d} \end{aligned} \tag{14.19}$$

has no solution with each $E_{j_1}^{(n_1)}(x,\omega)$, $E_{j_2}^{(n_2)}(x+m_1\omega,\omega)$, and $E_{j_3}^{(n_3)}(x+m_2\omega,\omega)$ being the evaluation of a segment with the parameters δ_0, δ_1 as in (a), (b) above.

Proof. For each

$$\omega \in \mathbb{T}_{c,a} \setminus \widetilde{\Omega}_n, \qquad \widetilde{\Omega}_n = \bigcup_{n \leq n' \leq 100n} \Omega_{n'}$$

we enumerate all possible segments as follows: the set

$$[-C(V), C(V)] \setminus \bigcup_{n \leq n' \leq 100n} \mathcal{E}_{n',\omega}$$

can be written as the union of no more than $e^{2n^{\delta_1}}$ intervals $(\underline{E}, \overline{E})$ of lengths $e^{-n^{\delta_1}}$ (with n large). Fixing such an $(\underline{E}, \overline{E})$ one obtains no more than $e^{2n^{\delta_1}}$ many I-segments $\left\{E_j^{(n')}(\cdot,\omega), \underline{x}, \bar{x}\right\}$ with $n \leq n' \leq 100n$ and $I = (\underline{E}, \overline{E})$. In total, there are no more than $e^{4n^{\delta_1}}$ many segments in this enumeration, each of

[9] In what follows, the parameter "d" is a large number that has nothing to do with the parameter appearing earlier in this section in connection with the rectangle $\mathcal{R}$.

which has slope bounded below by $s_0 := e^{-n^{2\delta_1}}$. Fix three n' in the specified range and denote them by n_1, n_2, n_3. In addition, fix three segments from our list which we denote by

$$\Big\{E^{(n_1)}_{j_1}(\cdot,\omega), \underline{x}_1, \bar{x}_1\Big\}, \qquad \Big\{E^{(n_2)}_{j_2}(\cdot,\omega), \underline{x}_2, \bar{x}_2\Big\}, \qquad \Big\{E^{(n_3)}_{j_3}(\cdot,\omega), \underline{x}_3, \bar{x}_3\Big\}$$

Let $C(V)$ be a large enough. The functions

$$\begin{aligned} f(x,\omega) &:= C(V)^{-1}(E^{(n_2)}_{j_2}(x+m_1\,\omega,\omega) - E^{(n_1)}_{j_1}(x,\omega)) \\ g(x,\omega) &:= C(V)^{-1}(E^{(n_3)}_{j_3}(x+m_2\,\omega,\omega) - E^{(n_1)}_{j_1}(x,\omega)) \end{aligned} \tag{14.20}$$

are defined on rectangles $\mathcal{R}_f := (x_1,x_2)\times(\omega_1,\omega_2)$ and $\widetilde{\mathcal{R}}_g := (x_1,x_2)\times(\omega_1',\omega_2')$, respectively, where

$$x_2 - x_1 \gtrsim \lambda_0, \quad \omega_2-\omega_1 \gtrsim \lambda_1/m_1, \quad \omega_2'-\omega_1' \gtrsim \lambda_1/m_2$$

with $\lambda_0 := e^{-n^{\delta_1}}$, $\lambda_1 := e^{-n^{3\delta_1}}$. This is a consequence of the stability of the segments under perturbations in x, ω.

Via an obvious covering argument, the total number of such rectangles $\mathcal{R}_f$ and $\widetilde{\mathcal{R}}_g$ that we need to consider is no larger than $(\lambda_0\lambda_1)^{-1}m_1$ and $(\lambda_0\lambda_1)^{-1}m_2$, respectively, with m_1, m_2 fixed (and up to multiplicative constants). Similarly, the number of choices of f which we need to consider with m_1 fixed is no larger than $e^{4n^{\delta_1}}(\lambda_0\lambda_1)^{-1}m_1$ and that of all possible g is no larger than $e^{4n^{\delta_1}}(\lambda_0\lambda_1)^{-1}m_2$. Finally, summing over all admissible choices of m_1, m_2 as in the statement yields

$$F \lesssim e^{4n^{\delta_1}}(\lambda_0\lambda_1)^{-1}\exp(2(\log N)^{1/4}), \quad G \lesssim e^{4n^{\delta_1}}(\lambda_0\lambda_1)^{-1}N^2 \tag{14.21}$$

where F, G denote the total number of f and g, respectively, that need to be considered in this enumeration. Note that by our choice of δ_1, one has

$$e^{n^{10\delta_1}} \le \exp\big((\log N)^{\frac12}\big) \le N^{\varepsilon}$$

for any $\varepsilon > 0$ provided N is large. We now verify the conditions of Proposition 14.7 for such a fixed choice of f, g living on some pair $\mathcal{R} = \mathcal{R}_f$, $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_g$ as above. In the notation of that proposition,

$$m_1 \gtrsim \mu \gtrsim m_1 s_0 \gtrsim \sqrt{m_1}, \quad m_2 \gtrsim \bar{\mu} \gtrsim m_2 s_0 \gtrsim \sqrt{m_2}, \quad K \lesssim 1, \quad r_0 = e^{-n^{2\delta_1}} = s_0$$

where we used that $m_1, m_2 \gg s_0^{-1}$. Since $r_0\mu \gg 1$, it follows that $r = \min(r_0^3\mu^2, r_0)/8 \asymp r_0 = s_0$. Moreover, due to $r_0 \le q \lesssim 1$, one has $\frac{q}{5r} \gtrsim 1$ so that the condition on σ_1 in Proposition 14.7 turns into the harmless $0 < \sigma_1 \lesssim 1$. We choose $h_1 := (\omega_2-\omega_1)/8 \gtrsim \lambda_1/m_1$ which implies that $\kappa_1 = h_1\mu K^{-1} \gtrsim h_1 m_1 s_0 \gtrsim \lambda_1 s_0$. Now set $\sigma_1 = N^{-2\varepsilon_0}$ and $\sigma_2 = N^{-d}$ which is admissible for Proposition 14.7 by the preceding. Then the constant M from (14.15) satisfies

$$M \lesssim \lambda_1^{-1}s_0^{-1}\log(s_0^{-1}\sigma_1^{-1}) \lesssim \lambda_1^{-2}\log(\sigma_1^{-1})$$

and our main condition on $\bar{\mu}$ reduces to

$$\bar{\mu} \gg \sigma_1^{-1}\lambda_1^{-2}\log(\sigma_1^{-1})$$

This holds because

$$\bar{\mu} \gtrsim m_2 s_0 \gg \sigma_1^{-2}\lambda_1^{-2}$$

Finally, since

$$\sigma_2 \ll \sigma_1 M^{-1}$$

if follows that the first bound in part (ii) of Proposition 14.7 reduces to

$$\sum_{\ell} |U_\ell| \lesssim \sigma_1$$

Applying Proposition 14.7, we define $\mathcal{B}_n'$ and $\mathcal{B}_n''$ as the union over all possible choices of f and g as explained above of all intervals U_ℓ and V_k respectively as in Proposition 14.7. Recall that due to property

(iii) in Proposition 14.7 the intervals U_ℓ depend on function f only. Therefore $\mathcal{B}'_n$ is a union of at most F intervals. The set $\mathcal{B}''_n$ is a union of at most FG intervals. Furthermore,

$$\begin{aligned}\mathrm{mes}\,(\mathcal{B}'_n) &\lesssim \sigma_1 F \le N^{-2\varepsilon_0} e^{4n^{\delta_1}} (\lambda_0\lambda_1)^{-1} \exp(2(\log N)^{1/4}) \lesssim N^{-\varepsilon_0}\\ \mathrm{mes}\,(\mathcal{B}''_n) &\lesssim \sigma_2 FG \lesssim N^{-d} e^{8n^{\delta_1}} (\lambda_0\lambda_1)^{-2} N^{2+2\varepsilon_0} \lesssim N^{-d+3}\end{aligned}$$

as claimed. The complexity bounds are as follows:

$$\begin{aligned}\mathrm{compl}(\mathcal{B}'_n) &\lesssim MF \lesssim \exp(\sqrt{N})\\ \mathrm{compl}(\mathcal{B}''_n) &\lesssim MFG \lesssim N^3\end{aligned}$$

as desired. Finally, suppose (14.19) had a solution for some x satisfying all the conditions stated above. Although the segments from (14.19) do not necessarily belong to our list of segments described in the beginning of the proof, locally around x they would have to agree with some choice of segment from our list. Therefore, for some choice of f, g as in (14.20) necessarily

$$|f(x,\omega)| < \sigma_2, \quad |g(x,\omega)| < \sigma_2$$

contradicting that $\omega \notin \mathcal{B}_n$, see Proposition 14.7. □

Remark 14.9. *The purpose of this remark is to comment further on the set of exceptional ω in Proposition 15.7. Recall that we assume that*

$$\omega \in \mathbb{T}_{c,a} \setminus \mathcal{B}(N), \quad \mathcal{B}(N) := \mathcal{B}'_n \cup \mathcal{B}''_n \cup \tilde{\Omega}_n \tag{14.22}$$

with the effective bounds

$$\begin{aligned}\mathrm{mes}\,(\mathcal{B}'_n) &< N^{-\varepsilon_0}, \quad \mathrm{compl}(\mathcal{B}'_n) \le \exp((\log N)^{1/2})\\ \mathrm{mes}\,(\mathcal{B}''_n) &< N^{-d+3}, \quad \mathrm{compl}(\mathcal{B}''_n) \le N^3\end{aligned}$$

Here $0 < \varepsilon_0 \ll 1$, $d \ge 4$ are arbitrary, provided

$$N \ge N_0(V, c, a, \gamma, \delta_0, \delta_1, A, d, \varepsilon_0)$$

and δ_0, δ_1, A are parameters as specified in Corollary 14.8. For the rest of this paper we fix all the parameters except d in such a way that the corollary holds. Thus the statement of Corollary 14.8 holds as long as N is large enough depending on d. Recall also that

$$\widetilde{\Omega}_n := \bigcup_{n \le n' \le 100n} \Omega_{n'}$$

where

$$\mathrm{mes}\,(\Omega_m) \le \exp(-(\log m)^A), \quad \mathrm{compl}(\Omega_m) \le m^C$$

$n \asymp (\log N)^A$*. It is convenient to use the outer Hausdorff measures $\mathcal{H}^\alpha_r(\mathcal{F})$, $\mathcal{F} \subset \mathbb{R}$*

$$\mathcal{H}^\alpha_r(\mathcal{F}) := \inf\Big\{ \sum_j |I_j|^\alpha \;:\; \mathcal{F} \subset \bigcup_j I_j,\ \sup_j |I_j| \le r \Big\}$$

where $0 < \alpha \le 1$, $r > 0$ are arbitrary. For $d > 7$ define

$$\alpha(d) = 4/(d-3), \quad r(N) = \exp(-(\log\log N)^A) \tag{14.23}$$

Then

$$\mathcal{H}^{\alpha(d)}_{r(N)}(\mathcal{B}(N)) \le \exp(-(\log\log N)^B)$$

where $B = A/2$.

15. Resonances and the formation of pre-gaps

The main objective of this section is to establish the resonance splitting picture for the Rellich parametrization of the eigenvalues similar to the one described in Figure 2. For ease of notation, we mostly drop ω from functions when it appears as an independent variable.

We begin with the following statement, which formalizes the idea that we can make a positive slope I-segment intersect with a negative of the same scale by means of a shift ot the form $m\omega$. Of crucial importance is the fact that the intersecting segments can be chosen to be regular unless we are inside a spectrum free interval of energies in the sense of Section 12.

Lemma 15.1. (i) *Fix $\delta > 0$ small and let $\{E^{(\underline{N})}_{j_1}(x), \underline{x}_1, \overline{x}_1\}$ and $\{E^{(\underline{N})}_{j_2}(x), \underline{x}_2, \overline{x}_2\}$ be a positive-slope and a negative-slope I-segment, respectively, where $I = [\underline{E}, \overline{E}]$, with $\overline{E} - \underline{E} > \exp(-\underline{N}^\delta)$. Then for all $\underline{N} \ge N_0(\delta)$ there exists an integer $m \in \big[\exp(\underline{N}^\delta), \exp(2\underline{N}^\delta)\big]$ and $x_0 \in (\underline{x}_1, \overline{x}_1)$ such that*

$$E^{(\underline{N})}_{j_1}(x_0) = E^{(\underline{N})}_{j_2}(x_0 + m\omega). \tag{15.1}$$

Moreover[10],

$$\operatorname{dist}(x_0, \{\underline{x}_j, \overline{x}_j\}) > C(V)^{-1}(\overline{E} - \underline{E}) \tag{15.2}$$

for $j = 1, 2$.

(ii) *Given a scale ℓ and interval (E_0', E_0''), $E_0'' - E_0' \ge \exp(-(\log \ell)^A)$, either the interval*

$$\Big(E_0' + \frac{1}{4}\exp(-(\log \ell)^A), E_0'' - \frac{1}{4}\exp(-(\log \ell)^A)\Big)$$

is spectrum free, or at some scale $\ell^2 \le \underline{N} \le \ell^{10}$ there exist a regular positive-slope I-segment $\{E^{(\underline{N})}_{j_1}(x), \underline{x}_1, \overline{x}_1\}$, a regular negative-slope I-segment $\{E^{(\underline{N})}_{j_2}(x), \underline{x}_2, \overline{x}_2\}$, $I = \big[\underline{E}, \overline{E}\big] \subset [E_0', E_0'']$, $\overline{E} - \underline{E} > \exp(-\underline{N}^\delta)$, an integer $m \in [\exp(\underline{N}^\delta), \exp(2\underline{N}^\delta)]$ and a point $x_0 \in (\underline{x}_1, \overline{x}_1)$ such that conditions (15.1) *and* (15.2) *hold.*

Proof. (i) Assume for instance, $\underline{x}_1 < \underline{x}_2$. Then necessarily also $\overline{x}_1 < \overline{x}_2$. Let $y_1 = \overline{x}_2 - \overline{x}_1$ and $y_2 = \underline{x}_2 - \underline{x}_1$. The function

$$h(E) = \big(E^{(\underline{N})}_{j_2}\big)^{-1}(E) - \big(E^{(\underline{N})}_{j_1}\big)^{-1}(E)$$

is strictly decreasing and satisfies $h(\overline{E}) = y_1$, $h(\underline{E}) = y_2$. Let $\Delta E := \overline{E} - \underline{E}$, $\overline{E}' = \overline{E} - \Delta E/4$ and $\underline{E}' = \underline{E} + \Delta E/4$, and define $y_1' := h(\overline{E}')$, $y_2' := h(\underline{E}')$. Then

$$y_2' - y_1' > C(V)^{-1}(\overline{E} - \underline{E}) > C(V)^{-1}\exp(-\underline{N}^\delta)$$

Hence, by the Diophantine nature of ω, there exists $\exp(\underline{N}^\delta) \le m \le \exp(2\underline{N}^\delta)$ so that $\{m\omega\} \in (y_1', y_2')$. Consequently, there is a unique $E_0 \in (\underline{E}, \overline{E})$ so that $h(E_0) = \{m\omega\}$. Set $x_0 := \big(E^{(\underline{N})}_{j1}\big)^{-1}(E_0)$. By construction, $\underline{x}_1 < x_0 < \overline{x}_1$ and

$$E^{(\underline{N})}_{j_2}(x_0 + m\omega) = E_0 = E^{(\underline{N})}_{j_1}(x_0)$$

as desired. Moreover, (15.2) follows from

$$\operatorname{dist}(E_0, \{\overline{E}, \underline{E}\}) > \Delta E/4$$

(ii) This part follows from part (i) due to Proposition 12.15. $\square$

For the rest of this section we fix a regular positive-slope I-segment $\{E^{(\underline{N})}_{j_1}(x), \underline{x}_1, \overline{x}_1\}$, a regular negative-slope I-segment $\{E^{(\underline{N})}_{j_2}(x), \underline{x}_2, \overline{x}_2\}$, $I = \big[\underline{E}, \overline{E}\big]$, $\overline{E} - \underline{E} > \exp(-\underline{N}^\delta)$,

(i) $|\partial_x E^{(\underline{N})}_{j_s}| \ge \exp(-\underline{N}^\delta)$ for any $x \in (\underline{x}_s, \overline{x}_s)$, $s = 1, 2$

[10] $\{\underline{x}_j, \overline{x}_j\}$ here is the set with these two points as elements.

(ii) $\overline{x}_s - \underline{x}_s \geq \exp(-\underline{N}^{\delta})$

where $\delta > 0$ can be made arbitrarily small but fixed and N large depending on δ. We also fix an integer $m \in [\exp(\underline{N}^{\delta}), \exp(2\underline{N}^{\delta})]$ and a point $x_0 \in (\underline{x}_1, \overline{x}_1)$ such that condition (15.1) holds. As usual, we denote the eigenvalues of $H_{[-N,N]}(x,\omega)$ by $E_j^{(N)}(x)$, $1 \leq j \leq 2N+1$, and a normalized eigenfunction corresponding to $E_j^{(N)}(x)$ by $\psi_j^{(N)}(x,\cdot)$. As in the proof of the previous lemma, $E_0 := E_{j_1}^{(\underline{N})}(x_0)$. In the following proposition, we pass to a larger scale. The idea is as follows: since they are *regular* the segments $E_{j_1}^{(\underline{N})}(\cdot)$ and $E_{j_1}^{(\underline{N})}(\cdot + m\omega)$ correspond to eigenfunctions supported on $[-\underline{N}, \underline{N}]$, and $[m-\underline{N}, m+\underline{N}]$, respectively which are exponentially small near the edges of these intervals. Hence, they each generate an approximate eigenstate of the operator $H_{[-N,N]}(x,\omega)$ with eigenvalues close to E_0. Proposition 15.2 quantifies these qualitative properties. What remains to be done as far as Figure 2 is concerned is to show that there is a true bottom arc E^-, i.e., an arc that achieves its maximum in the interior of the interval on which it is defined.

Proposition 15.2. *Let $0 < \varepsilon_0 \leq \frac{1}{20}$ be arbitrary but fixed, and $0 < \delta \ll \frac{\varepsilon_0}{A}$ where $A = A(V,\gamma,a,c)$. Set $N := [\exp(\underline{N}^{6\delta})]$, $n := [(\log N)^A]$. We assume that*

$$\omega \in \mathbb{T}_{c,a} \setminus (\mathcal{B}_n' \cup \mathcal{B}_n'' \cup \tilde{\Omega}_n)$$

as in Corollary 14.8.

(a) *For any interval $[N', N'']$, with $n \leq N'' - N' \leq 100n$, $N^{10\varepsilon_0} \leq |N'| \leq 2N$ and any $|x - x_0| < N^{-d-1}$ where d is as in Corollary 14.8 one has*

$$\operatorname{spec}(H_{[N',N'']}(x)) \cap (E_0 - \kappa, E_0 + \kappa) = \emptyset$$

with $\kappa := N^{-d}$

(b) *For any $x_0 - N^{-d-1} \leq x \leq x_0 + N^{-d-1}$ there exists j_1', j_2' (possibly depending on x) such that*

$$|E_{j_1}^{(\underline{N})}(x) - E_{j_1'}^{(N)}(x)| \leq \exp(-\underline{N}^{1/3}), \qquad |E_{j_2}^{(\underline{N})}(x+m\omega) - E_{j_2'}^{(N)}(x)| \leq \exp(-\underline{N}^{1/3})$$

(c) *For any $x_0 - N^{-d-1} \leq x_- \leq x_0 - e^{-\underline{N}^{\frac{1}{3}}}$ (resp. $x_0 + e^{-\underline{N}^{\frac{1}{3}}} \leq x_+ \leq x_0 + N^{-d-1}$) there exist j_-', j_-'', respectively j_+', j_+'', (possibly depending on x_-, x_+) such that*

$$E_0 - C(x_0 - x_-) < E_{j_-'}^{(N)}(x_-) < E_0 - (x_0 - x_-)\exp(-\underline{N}^{\delta})$$

$$E_0 - C(x_+ - x_0) < E_{j_-''}^{(N)}(x_+) < E_0 - (x_0 - x_+)\exp(-\underline{N}^{\delta})$$

$$E_0 + (x_0 - x_-)\exp(-\underline{N}^{\delta}) < E_{j_+'}^{(N)}(x_-) < E_0 + C(x_0 - x_-)$$

$$E_0 + (x_+ - x_0)\exp(-\underline{N}^{\delta}) < E_{j_+''}^{(N)}(x_+) < E_0 + C(x_+ - x_0)$$

(d) *If $E_j^{(N)}(x) \in (E_0 - N^{-d}/2, E_0 + N^{-d}/2)$ for some $x \in [x_0 - N^{-d-1}, x_0 + N^{-d-1}]$ then*

$$\left|\psi_j^{(N)}(x,n')\right| \leq \exp(-\gamma|n'|/4) \qquad \forall\, |n'| > N^{\frac{1}{2}}$$

(e) *If $E_j^{(N)}(x), E_{j'}^{(N)}(x) \in (E_0 - N^{-d}/2, E_0 + N^{-d}/2)$, $j \neq j'$, $x \in [x_-, x_+]$ then*

$$|E_j^{(N)}(x) - E_{j'}^{(N)}(x)| \geq \tau = \exp(-N^{11\varepsilon_0})$$

Proof. To prove (a) we use Corollary 14.8 Recall that we assume that $\omega \in \mathbb{T}_{c,a} \setminus (\mathcal{B}_n' \cup \mathcal{B}_n'' \cup \tilde{\Omega}_n)$. We set $\delta_0 := \delta$, $\delta_1 := 3\delta$, $\varepsilon_0 := 1/50$, $n_1 := \underline{N}$, $n_2 = n_1$, $n_3 := N'' - N' + 1$, $m_1 := m$, $m_2 := N'$. Then all conditions of Corollary 14.8 hold. Let $|x - x_0| \leq N^{-d-1}$. Then

$$|E_{j_1}^{(\underline{N})}(x) - E_{j_2}^{(\underline{N})}(x + m_1\omega)| \leq C(V)|x - x_0| \leq N^{-d}$$

Due to Corollary 14.8 one has $|E_{j_1}^{(\underline{N})}(x) - E_j^{(n_3)}(x + m_2\omega)| \geq N^{-d}$ for any $1 \leq j \leq 2n_3 + 1$ which proves (a). Since $\{E_{j_1}^{(\underline{N})}(x), \underline{x}_1, \overline{x}_1\}$ is a regular segment we have $\underline{N}^{1/2} \leq \nu_{j_1}^{(\underline{N})}(x) \leq \underline{N} - \underline{N}^{1/2}$. Therefore,

$$\|(H_{[-N,N]}(x) - E_{j_1}^{(\underline{N})}(x))\psi_{j_1}^{(\underline{N})}(x, \cdot)\| \leq \exp(-\underline{N}^{1/3})$$

whence there exists j_1' such that

$$|E_{j_1}^{(\underline{N})}(x) - E_{j_1'}^{(N)}(x)| \leq \exp(-\underline{N}^{1/3})$$

The proof of the statement regarding $E_{j_2}^{(\underline{N})}(x)$ is similar. Thus (b) holds. Because of (15.2) and

$$\begin{aligned} &\partial_x E_{j_1}^{(\underline{N})}(x) > \exp(-\underline{N}^{\delta}),\ x \in [\underline{x}_1, \bar{x}_1] \\ &\partial_x E_{j_2}^{(\underline{N})}(x) < \exp(-\underline{N}^{\delta}),\ x \in [\underline{x}_2, \bar{x}_2] \end{aligned} \tag{15.3}$$

part (c) follows from (b). To validate (d) and (e) we invoke Corollary 7.2. Indeed, part (a) of the current proposition implies that the condition needed for Corollary 7.2 are met. Therefore, (d) and (e) hold. □

Remark 15.3. *The previous proposition is based on the elimination via Corollary 14.8 and not via resultants as in Proposition 5.5. This is crucial, as the latter method of elimination requires the removal of some subset of the energy E. Although the subset in Proposition 5.5 is small, its removal would destroy the argument.*

The following lemma establishes the existence of the E^- branch in Figure 2.

Lemma 15.4. *For N large, there exists j_0 with*

$$\left|E_{j_0}^{(N)}(x_0) - E_0\right| < C(V)N^{-d-4} \tag{15.4}$$

and an interval $[x_-, x_+]$,

$$0 < x_0 - x_- \leq N^{-d-4}, \qquad 0 < x_+ - x_0 \leq N^{-d-4} \tag{15.5}$$

such that $E_{j_0}^{(N)}(x)$ assumes its maximum over the interval $[x_-, x_+]$ at some point $x^{(0)}$ which is located "properly inside the interval", i.e.,

$$x_- + C^{-1}N^{-d-13} < x^{(0)} < x_+ - C^{-1}N^{-d-13} \tag{15.6}$$

Proof. To establish this lemma we use Proposition 15.2 and the property that the graphs of $E_j^{(N)}(x)$, $1 \leq j \leq 2N+1$ never intersect. Set

$$\begin{aligned} &x_-(k) = x_0 - kN^{-d-8},\ \ 1 \leq k \leq N^4 \\ &E_-(k) = E_{j_1}^{(\underline{N})}(x_-(k)) \\ &J_-(k) = \left\{j : \left|E_j^{(N)}(x_-(k)) - E_-(k)\right| < \exp(-\underline{N}^{\frac{1}{3}})\right\} \end{aligned}$$

where $k \geq 1$. Due to part (b) in Proposition 15.2 each set $J_-(k)$ is non-empty. Furthermore, due to the fact $\partial_x E_{j_1}^{(\underline{N})}(x) > \exp(-\underline{N}^{\delta})$ one also has

$$E_-(k-1) > E_-(k) + N^{-d-8}\exp(-\underline{N}^{\delta})$$

Let $j \in J_-(k)$ for some k. Assume that

$$\max\left\{E_j^{(N)}(x)\ :\ x \in [x_-(k), x_0]\right\} < E_-(k-N) - \exp(-\underline{N}^{1/3}) \tag{15.7}$$

Then clearly one has

$$j \notin J_-(k') \text{ for any } k' < k - N \tag{15.8}$$

Since there are only $2N+1$ eigenvalues $E_j^{(N)}(x)$ in total there exist $N^3 \geq k_- \geq N^3/2$ and $j_- \in J_-(k_-)$ such that

$$\max\left\{E_{j_-}^{(N)}(x)\ :\ x \in [x_-(k_-), x_0]\right\} \geq E_-(k_- - N) - \exp(-\underline{N}^{1/3})$$

Analogously, we set

$$\begin{aligned} x_+(k) &:= x_0 + kN^{-d-11}, \ \ 1 \le k \le N^7, \\ E_+(k) &:= E^{(\underline{N})}_{j_2}(x_+(k) + m\omega), \\ J_+(k) &:= \big\{ j \ : \ |E^{(N)}_j(x_+(k)) - E_+(k)| < \exp(-\underline{N}^{\frac{1}{3}}) \big\} \end{aligned}$$

We want to show that there exist k_+ and $j_+ \in J_+(k_+)$ such that the following conditions hold

(α) $\max\big\{E^{(N)}_{j_+}(x) : x \in [x_0, x_+(k_+)]\big\} \ge E_+(k_+ - N) - \exp(-\underline{N}^{\frac{1}{3}})$,

(β) $E_-(k_-) - N^{-d-11} < E_+(k_+) < E_-(k_-) + CN^{-d-8}$.

To achieve this note that due to (15.3) and $k_- > N^3/2$, one has

$$E_-(k_-) < E_0 - \frac{1}{2} N^{-d-5} \exp(-\underline{N}^{\delta})$$

On the other hand,

$$E_-(k_-) \ge E_-(N^3) > E_0 - CN^{-d-5}$$

Once again, due to (15.3),

$$E^{(\underline{N})}_{j_2}(x_+(N^7) + m\omega) < E_0 - N^{-d-4} \exp(-\underline{N}^{\delta})$$

Hence, there exists $k'_+ \le N^7$ such that $E_+(k'_+ + 1) \le E_-(k_-) < E_+(k'_+)$. Since

$$E_0 - \frac{1}{2} N^{-d-5} \exp(-\underline{N}^{\delta}) \ge E_-(k_-) \ge E^{(\underline{N})}_{j_2}(x_+(k'_+ + 1) + m\omega) > E_0 - Ck'_+ N^{-d-11}$$

one has $N^7 \ge k'_+ \gtrsim N^6 \exp(-\underline{N}^{\delta})$. Applying an argument analogous to the one used to determine k_-, one finds $k_+ \in [k'_+ - N^3, k'_+]$ and $j_+ \in J_+(k_+)$ such that condition (α) holds. Furthermore,

$$\begin{aligned} E_+(k_+) &> E_+(k'_+) > E_-(k_-) \\ E_+(k_+) &< E_+(k'_+ - N^3) < E_+(k'_+) + CN^{-d-8} \\ &< E_+(k'_+ + 1) + CN^{-d-11} + CN^{-d-8} \\ &< E_-(k_-) + CN^{-d-11} + CN^{-d-8} \end{aligned}$$

Therefore, condition (β) holds. To finish the proof of the lemma assume for instance that

$$E^{(N)}_{j_-}(x_+(k_+)) \le E^{(N)}_{j_+}(x_+(k_+)) \tag{15.9}$$

Then

$$\begin{aligned} &\max\big\{E^{(N)}_{j_-}(x) \ : \ x \in [x_+(k_+) - N^{-d-11}, x_+(k_+)]\big\} \\ &\le E^{(N)}_{j_-}(x_+(k_+)) + CN^{-d-11} \le E^{(N)}_{j_+}(x_+(k_+)) + CN^{-d-11} \\ &\le E_+(k_+) + \exp(-\underline{N}^{\frac{1}{3}}) + CN^{-d-11} \\ &\le E^{(N)}_{j_-}(x_-(k_-)) + 3CN^{-d-8} \end{aligned}$$

Furthermore,

$$\max\big\{E^{(N)}_{j_-}(x) \ : \ x \in [x_-(k_-), x_-(k_-) + N^{-d-8}]\big\} \le E^{(N)}_{j_-}(x_-(k_-)) + CN^{-d-8}$$

On the other hand,

$$\begin{aligned} &\max\big\{E^{(N)}_{j_-}(x) \ : \ x \in [x_-(k_-), x_+(k_+)]\big\} \ge \max\big\{E^{(N)}_{j_-}(x) \ : \ x \in [x_-(k_-), x_0]\big\} \\ &\ge E_-(k_- - N) - \exp(-\underline{N}^{1/3}) \ge E^{(N)}_{j_-}(x_-(k_-)) + N^{-d-7} \exp(-\underline{N}^{\delta}) - 2\exp(-\underline{N}^{1/3}) \\ &\ge E^{(N)}_{j_-}(x_-(k_-)) + \frac{1}{2} N^{-d-7} \exp(-\underline{N}^{\delta}) \end{aligned}$$

Set $j_0 = j_-$, $[x_-, x_+] := [x_-(k_-), x_+(k_+)]$. Then (15.6) holds. Note that

$$0 < x_0 - x_-(k_-) \le N^{-d-4}, \quad 0 < x_+(k_+) - x_0 \le N^{-d-4}$$

since $k_- \le N^4$, $k_+ \le N^7$ as well as

$$\begin{aligned} |E_{j_-}^{(N)}(x_0) - E_0| &\le |E_{j_-}^{(N)}(x_0) - E_{j_-}^{(N)}(x_-(k_-))| + |E_{j_-}^{(N)}(x_-(k_-)) - E_-(k_-)| + |E_-(k_-) - E_0| \\ &\le C(V)N^{-d-4} \end{aligned}$$

Hence (15.4), (15.5) also hold. Consider now the case opposite to (15.9). Since the graphs of $E_{j_-}^{(N)}(x)$ and $E_{j_+}^{(N)}(x)$ do not intersect one has in this case

$$E_{j_+}^{(N)}(x) < E_{j_-}^{(N)}(x)$$

for all x. Hence,

$$\begin{aligned} &\max\bigl\{E_{j_+}^{(N)}(x) : x \in [x_-(k_-), x_-(k_-) + N^{-d-13}]\bigr\} \le \max\bigl\{E_{j_-}^{(N)}(x) : x \in [x_-(k_-), x_-(k_-) + N^{-d-13}]\bigr\} \\ &\le E_{j_-}^{(N)}(x_-(k_-)) + CN^{-d-12} \le E_-(k_-) + \exp(-\underline{N}^{1/3}) + CN^{-d-12} \le E_+(k_+) + 2N^{-d-11} \\ &\le E_{j_+}^{(N)}(x_+(k_+)) + \exp(-\underline{N}^{1/3}) + 2N^{-d-11} \le E_{j_+}^{(N)}(x_+(k_+)) + 3N^{-d-11} \end{aligned}$$

Furthermore,

$$\max\bigl\{E_{j_+}^{(N)}(x) : x \in [x_+(k_+) - N^{-d-13}, x_+(k_+)]\bigr\} \le E_{j_+}^{(N)}(x_+(k_+)) + CN^{-d-13}$$

On the other hand,

$$\begin{aligned} &\max\bigl\{E_{j_+}^{(N)}(x) : x \in [x_-(k_-), x_+(k_+)]\bigr\} \ge \max\bigl\{E_{j_+}^{(N)}(x) : x \in [x_0, x_+(k_+)]\bigr\} \ge \\ &E_+(k_+ - N) - \exp(-\underline{N}^{1/3}) \ge E_+(k_+) + N^{-d-10}\exp(-\underline{N}^{\delta}) - \exp(-\underline{N}^{1/3}) \ge \\ &E_{j_+}^{(N)}(x_+(k_+)) + N^{-d-10}\exp(-\underline{N}^{\delta}) - 2\exp(-\underline{N}^{1/3}) \ge E_{j_+}^{(N)}(x_+(k_+)) + \frac{1}{2}N^{-d-10}\exp(-\underline{N}^{\delta}) \end{aligned}$$

Set $j_0 = j_+$. Thus (15.6) and (15.5) follow. Finally,

$$\begin{aligned} |E_{j_+}^{(N)}(x_0) - E_0| &\le |E_{j_+}^{(N)}(x_0) - E_{j_+}^{(N)}(x_+(k_+))| + |E_{j_+}^{(N)}(x_+(k_+)) - E_+(k_+)| + |E_+(k_+) - E_0| \\ &\le C(V)N^{-d-4} \end{aligned}$$

and (15.4) holds as claimed. □

For convenience we summarize the conclusions of Lemma 15.4 and Proposition 15.2 in the following corollary.

Corollary 15.5. *Using the notations of Proposition 15.2 there exists j_0 and an interval $[x_-, x_+] \subset [x_0 - N^{-d-4}, x_0 + N^{-d-4}]$ such that*

(1) *The function $E^{(-)}(x) := E_{j_0}^{(N)}(x), x \in [x_-, x_+]$ assumes its maximal value of the interval $[x_-, x_+]$ at some point $x^{(0)} \in [x_- + CN^{-d-13}, x_+ - CN^{-d-13}]$*
(2) *For any $x \in [x_-, x_+]$ the operator $H_{[-N,N]}(x)$ has no eigenvalues in the interval*

$$(E^{(-)}(x), E^{(-)}(x) + \tau), \quad \tau = \exp(-N^{11\varepsilon_0})$$

(3) *The eigenfunction $\psi^{(-)}(x, n') := \psi_{j_0}^{(N)}(x, n')$, obeys $|\psi^{(-)}(x, n')| \le \exp(-\frac{\gamma}{4}|n'|)$, for $|n'| \ge N^{1/2}$.*

Next, we will establish that any energy $E \in (E^{(-)}(x^{(0)}), E^{(-)}(x^{(0)}) + \tau)$ generates two complex zeros of $f_N(\cdot, E)$ with imaginary part bounded below. In order to do this, we apply the Weierstrass preparation theorem as in Corollary 2.28 to $f_N(z, E)$ relative to the z-variable locally around the point

$$(z^{(0)}, E^{(0)}), \quad z^{(0)} = e(x^{(0)}), \quad E^{(0)} = E^{(-)}(x^{(0)})$$

Thus, there exist a polynomial $P(z,E) = z^k + a_{k-1}(E)z^{k-1} + \cdots + a_0(E)$ with $a_j(E)$ analytic in $\mathcal{D}(E^{(0)}, r_1)$, where $r_1 \asymp \exp(-N^{12\varepsilon_0})$, and an analytic function $g(z,E)$, $(z,E) \in \mathcal{P} = \mathcal{D}(z^{(0)}, r_1) \times \mathcal{D}(E^{(0)}, r_1)$ such that:

(a) $f_N(z,E) = P(z,E)g(z,E)$,
(b) $g(z,E) \neq 0$ for any $(z,E) \in \mathcal{P}$,
(c) For any $E \in \mathcal{D}(E_0, r_1)$, the polynomial $P(.,E)$ has no zeros in $\mathbb{C}\backslash\mathcal{D}(z^{(0)}, r_1)$,
(d) $1 \leq k = \deg P_N(\cdot, \omega, E) \leq (\log N)^{C_0}$

Here the property $k \geq 1$ is due to the fact that $f_N(z^{(0)}, E^{(0)}) = 0$. We can now derive the following result:

Lemma 15.6. *For any $E \in (E^{(0)} + r_1/4, E^{(0)} + r_1/2)$, the Dirichlet determinant $f_N(\cdot, E)$ has at least two complex zeros $\zeta^{\pm} = \zeta^{\pm}(E) = e(x(E) \pm iy(E)) \in \mathcal{D}(e(x^{(0)}), r_1)$, with $r_1/C_1 < |y(E)| < r_1$.*

Proof. For any $E \in \mathcal{D}(E_0, r_1)$, the polynomial $P(\cdot, E)$ has at least one zero $\zeta(E)$, with $\zeta(E) \in \mathcal{D}(z^{(0)}, r_1)$. Let $E_1 \in (E^{(0)} + r_1/4, E^{(0)} + r_1/2)$ be arbitrary, and let $\zeta_1 = \zeta(E_1) = e(x_1 + iy_1)$. Note that $E^{(0)} - Cr_1 \leq E^{(-)}(x_1) \leq E^{(0)}$. Therefore,

$$E^{(-)}(x_1) + r_1/4 \leq E^{(0)} + r_1/4 \leq E_1 \leq E^{(0)} + r_1/2 \leq E^{(-)}(x_1) + Cr_1 \leq E^{(-)}(x_1) + \tau/2$$

Furthermore, recall that due to condition (2) of Corollary 15.5, the operator $H_N(x_1)$ has no eigenvalues in the interval $(E^{(-)}(x_1), E^{(-)}(x_1) + \tau)$. In particular,

$$\operatorname{dist}\left[\operatorname{spec}\left(H_{[-N,N]}(x_1)\right), E_1\right] \geq r_1/4.$$

Since $H_N(x_1)$ is self adjoint and

$$\|H_{[-N,N]}(x_1) - H_{[-N,N]}(x_1 + iy_1)\| < C|y_1|,$$

one has also

$$0 = \operatorname{dist}\left[\operatorname{spec}\left(H_{[-N,N]}(x_1 + iy_1)\right), E_1\right] > r_1/4 - C|y_1|$$

The determinant $f_N(e(x), E_1)$ assumes only real values for real x. Therefore, each complex zero ζ_1 produces a conjugate zero and we are done. $\square$

The following is the main result of this section. Due to the fact that an eigenfunction ψ of $H_{[-N,N]}(x,\omega)$ which is very well localized in $[-N,N]$ remains close to an eigenfunction of this operator if it is translated inside of the interval we obtain a whole sequence of complex zeros as in the previous lemma. The parameters δ, ε_0 are as above.

Proposition 15.7. *Using the notations of part (ii) in Lemma 15.1, assume that*

$$\left(E_0' + \frac{1}{4}\exp(-(\log \ell)^A),\ E_0'' - \frac{1}{4}\exp(-(\log \ell)^A)\right)$$

is not spectrum-free. Then there exists $N = [\exp(\underline{N}^{6\delta})]$ with $\ell^2 \leq \underline{N} \leq \ell^{10}$ and an interval $(E', E'') \subset (E_0', E_0'')$, $E'' - E' = \exp(-N^{3\delta})$ such that for any $E \in (E', E'')$ the Dirichlet determinant $f_N(\cdot, \omega, E)$ has a sequence of zeros $\zeta_k^{\pm} = e(x_k \pm iy_k)$, where k runs in the interval $(-N + 2N^{1/2}, N - 2N^{1/2})$, with

$$\|x_k - x_0 - k\omega\| < \exp(-N^{1/8}), \quad \exp(-N^{13\varepsilon_0}) < |y_k| \leq \exp(-N^{10\varepsilon_0})$$

Proof. Due to part (a) of Proposition 15.2 one has

$$\operatorname{spec}\left(H_{[N',N'']}(x,\omega)\right) \cap (E_0 - \kappa, E_0 + \kappa) = \emptyset$$

for any $|x - x_0| \leq N^{-d-1}$ and any interval $[N', N'']$ with

$$n \leq N'' - N' \leq 100n, \quad N^{10\varepsilon_0} \leq |N'| \leq 2N$$

where n is as above, $\kappa := N^{-d}$. Therefore, due to Proposition 9.3, one obtains

$$\nu_{f_{[N_1,N_2]}(\cdot,E)}(e(x), R_0) = 0 \tag{15.10}$$

for any

$$[N_1, N_2] \subset [-2N, 2N] \setminus [-2N^{1/4}, 2N^{1/4}], \qquad N_2 - N_1 \geq n,$$

and any $E \in (E_0 - \kappa/2, E_0 + \kappa/2)$, where $R_0 =: \exp(-(\log N)^C)$. Let $x^{(0)}$, $E \in (E^{(0)} + r_1/2, E^{(0)} + r_1/2)$ be as in Lemma 15.6. Then the Dirichlet determinant $f_N(\cdot, E)$ has a complex zero

$$\zeta^+(E) = e(x(E) + iy(E)) \in \mathcal{D}(e(x^{(0)}), r_1)$$

with $y(E) > r_1/C_1$. Set $N_0 := [N^{1/3}]$. Then (15.10) implies in particular that N_0, $-N_0$ are adjusted to $(\mathcal{D}(\zeta^+(E), R_0), E^{(0)})$ at scale $\ell_0 := [N^{1/4}]$. Due to Proposition 9.3, with (15.10) taken into account, one obtains

$$\nu_{f_{[-N_0,N_0]}(\cdot,E)}(\zeta^+(E), r_0) \geq 1 \tag{15.11}$$

where $r_0 = \exp(-\ell_0^{1/2}) \asymp \exp(-N^{1/8})$. Let $k \in [-N + 2N^{1/2}, N - 2N^{1/2}]$ be arbitrary. Recall that

$$f_{[a,b]}(e(x + k\omega + iy), E) = f_{[a+k,b+k]}(e(x + iy), E)$$

for any x, y, E and any $[a, b]$. Using once again Proposition 9.3 with (15.10) and (15.11) taken into account yields

$$\nu_{f_{[-N,N]}(\cdot,E)}(\zeta^+(E)e(k\omega), r_0) \geq 1$$

Hence there exists $\zeta_k^+(E)$ as claimed. The argument for $\zeta_k^-(E)$ is similar. □

By a pre-gap we mean an interval of energies (E', E'') as described in Proposition 15.7. It remains to show that pre-gaps do not fill up again as we pass to larger scales. The following section makes this precise.

16. Proofs of Theorems 1.1, 1.2

To prove Theorem 1.2 we use Proposition 15.7 in combination with the analysis of the quantities

$$\mathcal{M}_N(E, R_1, R_2) = \tfrac{1}{N}\#\big\{z \in \mathcal{A}_{R_1,R_2} : f_N(z, \omega, E) = 0\big\} \tag{16.1}$$

from Section 4. Recall the following relations established for these quantities, cf. Lemma 4.6:

$$\begin{aligned} &\mathcal{M}_N(\omega, E, R_1 + r_2, R_2 - r_2) \leq \mathcal{M}_n(\omega, E, R_1 - r_2, R_2 + r_2) + n^{-1/4} \\ &\mathcal{M}_n(\omega, E, R_1 + r_2, R_2 - r_2) \leq \mathcal{M}_N(\omega, E, R_1 - r_2, R_2 + r_2) + n^{-1/4} \end{aligned} \tag{16.2}$$

for any $n > N_0(V, \gamma, a, c, \sigma)$, $N > \exp(\gamma\, n^\sigma)$, $1 - \rho^{(0)} < R_1 < R_2 < 1 + \rho^{(0)}$ where $N_0 = N_0(V, c, a, \gamma)$, $\rho^{(0)} = \rho^{(0)}(V, c, a, \gamma) > 0$, $r_2 = n^{-1/4}(R_2 - R_1)$, provided $r_2 > \exp(-\gamma\, n^\sigma/100)$. Here $\sigma > 0$ is small but fixed and γ stands for the lower bound on the Lyapunov exponent as usual. In what follows, $0 < \varepsilon_0 \ll 1$ is small and fixed as in the previous section and we will set $\sigma := 20\varepsilon_0$.

Lemma 16.1. *Using the notations of Proposition 15.7 one has for any $E \in (E', E'')$, and any $N_1 > \exp(\gamma\, N^{20\varepsilon_0})$*

$$\mathcal{M}_{N_1}(E, R'_-(N), R'_+(N)) \leq \mathcal{M}_N(E, R''_-(N), R''_+(N)) - 2 + CN^{-1/4} \tag{16.3}$$

where $R'_\pm(N) = 1 \pm \exp(-N^{14\varepsilon_0})$, $R''_\pm(N) = 1 \pm \exp(-N^{9\varepsilon_0})$

Proof. Due to Proposition 15.7 for any $E \in (E', E'')$, the Dirichlet determinant $f_N(\cdot, \omega, E)$ has a sequence of zeros $\zeta_k^\pm = e(x_k \pm iy_k)$, where k runs in the interval $(-N + 2N^{1/2}, N - 2N^{1/2})$,

$$\|x_k - x_0 - k\omega\| < \exp(-N^{1/8}), \quad \exp(-N^{13\varepsilon_0}) < |y_k| \leq \exp(-N^{10\varepsilon_0})$$

Since $\omega \in \mathbb{T}_{c,a}$, all these zeros are different, i.e., $\zeta_k^\pm \neq \zeta_{k_1}^\pm$ if $k \neq k_1$. Hence,

$$\mathcal{M}_N(E, R_{1,-}, R_{1,+}) \leq \mathcal{M}_N(E, R''_-(N), R''_+(N)) - 2 + 2N^{-1/2}$$

where $R_{1,\pm} = 1 \pm \exp(-2N^{13\varepsilon_0})$. Combining this estimate and (16.2) one obtains (16.3). □

We can now prove Theorems 1.2 and 1.1.

Proof of Theorem 1.2. Following the notations of Remark 14.9 given N we denote by $\mathcal{B}(N)$ the set of exceptional values of ω defined in Corollary 14.8. Recall that due to Remark 14.9 one has

$$\mathcal{H}^{\alpha(d)}_{r(N)}(\mathcal{B}(N)) \le \exp(-(\log\log N)^B)$$

where $\mathcal{H}^{\alpha}_r$ stands for the corresponding Hausdorff outer measure, $r(N) = \exp(-(\log\log N)^A)$, $\alpha(d) = 4/(d-3)$, $A, B > 1$ are constants, and $d > 7$ is arbitrary and provided $N \ge N_0(V, c, a, \gamma, d)$. We need to iterate the result of Lemma 16.1. Set with some $0 < \delta \ll \varepsilon_0$ as in the previous section, $\overline{N}(N,1) := N, \overline{N}(N,t+1) := [\exp((\overline{N}(N,t))^\delta)]$ for all $t \ge 1$, and

$$\mathcal{B}(N,T) := \bigcup_{1\le t\le T} \mathcal{B}(\overline{N}(N,t))$$

Assume that $\omega \in \mathbb{T}_{c,a} \setminus \mathcal{B}(N,T)$. Given arbitrary $1 \le t \le T$ and an interval $(E_{t,1}, E_{t,2})$ with

$$E_{t,2} - E_{t,1} \ge \exp(-(\log \overline{N}(N,t))^C),$$

either $(E_{t,1}, E_{t,2})$ contains a spectrum-free subinterval $(E'_{t,1}, E'_{t,2})$ with

$$E'_{t,2} - E'_{t,1} \ge \exp(-(\log \overline{N}(N,t))^C),$$

or there exists a subinterval (E'_t, E''_t) with

$$E''_t - E'_t \ge \exp(-\overline{N}(N,t+1)^{3\delta})$$

such that

$$\mathcal{M}_{\underline{N}(N,t+1)}(E, R'_-(\overline{N}(N,t)), R'_+(\overline{N}(N,t))) \le \mathcal{M}_N(E, R''_-(\overline{N}(N,t)), R''_+(\overline{N}(N,t))) - 2 + C\overline{N}(N,t)^{-1/4} \tag{16.4}$$

for any $E \in (E'_t, E''_t)$. Hence, given an arbitrary interval (E_1, E_2) with

$$E_2 - E_1 \ge \exp(-(\log N)^C),$$

either (E_1, E_2) contains a spectrum-free subinterval $(E'_{T,1}, E'_{T,2})$ with

$$E'_{T,2} - E'_{T,1} \ge \exp(-(\log \overline{N}(N,T))^C),$$

or there exists a subinterval (E'_T, E''_T) with

$$E''_T - E'_T \ge \exp(-\overline{N}(N,T+1)^{3\delta})$$

such that

$$\mathcal{M}_{\underline{N}(N,T+1)}(E, R'_-(\overline{N}(N,T)), R'_+(\overline{N}(N,T))) \le \mathcal{M}_N(E, R''_-(N), R''_+(N)) - 2T + CN^{-1/4} \tag{16.5}$$

for any $E \in (E'_T, E''_T)$. Recall that due to Lemma 11.5 on translations of regular I-segments one has in particular

$$\mathcal{M}_N(E, R_1, R_2) \ge 1 - 2N^{-1/2}$$

for any $E \in I$, and any R_1, R_2, provided there exists at least one regular segment $\{E^{(N)}_j(x,\omega), \underline{x}, \overline{x}\}$. On the other hand, $\mathcal{M}_N(E, R_1, R_2) \le C(V)$ for any N, R_1, R_2. In particular, using the notations of (16.3) with $T := [C(V)/2] + 1$ one has

$$\mathcal{M}_{\underline{N}(N,T+1)}(E, R'_-(\overline{N}(N,T)), R'_+(\overline{N}(N,T))) \le \mathcal{M}_N(E, R''_-(N), R''_+(N)) - 2T + CN^{-1/4} \lesssim N^{-1/4}$$

for any $E \in (E'_T, E''_T)$. Combining this estimate with (16.2) one obtains

$$\mathcal{M}_{N_1}(E, R'_-(\underline{N}(N,T+1)), R'_+(\underline{N}(N,T+1))) \le \mathcal{M}_N(E, R''_-(N), R''_+(N)) < 5N^{-1/4}$$

for any $N_1 \ge \underline{N}(N, T+2)$. Consequently, there is no regular I-segment

$$\{E^{(N_1)}_j(x,\omega), \underline{x}, \overline{x}\}, \qquad I \subset (E'_T, E''_T)$$

Due to Proposition 12.15 the interval (E'_T, E''_T) is spectrum-free. Finally, set $\mathcal{B}_{N,d} := \mathcal{B}(N,T)$. Then

$$\mathcal{H}^{\alpha(d)}_{r(N)}(\mathcal{B}_{N,d}) \le T + 1 \le C(V)$$

and we are done. □

Proof of Theorem 1.1. It is convenient to split the proof into a few steps. We enumerate these steps as a, b, c, and d.

(a) Using the notations of Proposition 13.1 assume that $\omega \in \mathbb{T}_{c,a}\setminus\hat{\Omega}_{N_1}$, $x \in \mathbb{T} \setminus \hat{\mathcal{B}}_{N_1,\omega}$ for some $N_1 \geq N_0$. Assume that (E_0', E_0'') is a spectrum free interval. Then for any N_2 there exists $N \geq N_2$ such that

$$\bigcup_{x\in\mathbb{T}} \mathrm{spec}\left(H^{(P)}_{[-N+1,N]}(x,\omega)\right) \cap (E_0', E_0'') = \emptyset$$

Let $E_0 := (E_0' + E_0'')/2$, $\sigma_0 := E_0'' - E_0'$. Then

$$\min\left\{|E_0 - E_j^{(N)}(x,\omega)| \ : \ E_j^{(N)}(x,\omega) \in (E', E''), \nu_j^{(N)}(x,\omega) \in [-N + N^{\frac12},\ N - N^{\frac12}]\right\} \geq \sigma_0/8 \tag{16.6}$$

Indeed, assume that $|E_0 - E_{j_0}^{(N)}(x,\omega)| \leq \sigma_0/4$ for some j_0 with

$$E_{j_0}^{(N)}(x,\omega) \in (E', E''), \quad \nu_{j_0}^{(N)}(x,\omega) \in [-N + N^{\frac12}, N - N^{\frac12}]$$

Then

$$\|(H^{(P)}_{[-N+1,N]}(x,\omega) - E_{j_0}^{(N)}(x,\omega))\psi_{j_0}^{(N)}(x,\omega,\cdot)\| \leq \exp(-\gamma N^{1/2}) \leq \sigma_0/4$$

Hence,

$$\mathrm{dist}\left[E_0, \mathrm{spec}\left(H^{(P)}_{[-N+1,N]}(x,\omega)\right)\right] \leq \sigma_0/2$$

contrary to our assumption. Thus (16.6) is valid. Due to Remark 13.5 for any $x' \in \mathbb{T}$ and any $|E - E_0| \leq \sigma_0/8$ one has $E \notin \mathrm{spec}(H(x',\omega))$.

(b) Assume that $\omega \in \mathbb{T}_{c,a}\setminus(\hat{\Omega}_{N_1} \cup \mathcal{B}_{N_1,d})$, for some $N_1 \geq N_0$, where $\mathcal{B}_{N_1,d}$ is defined as in Theorem 1.2. Let (E', E'') be an arbitrary interval. Then (E', E'') contains a spectrum free interval (E_0', E_0''). Then due to (a), (E_0', E_0'') contains an interval (E_1', E_1'') such that $\Sigma_\omega \cap (E_1', E_1'') = \emptyset$

(c) Set $\Omega' := \bigcap_{N\geq N_0} \hat{\Omega}_{N_1}$. Then Ω' has Hausdorff dimension zero. Furthermore, proceeding inductively, for each $d \geq 7$ pick an arbitrary $N(d)$ large enough so that $\mathcal{B}_{N(d),d}$ is defined, and also $N(d) \geq \max(\exp(\exp(d)), N(d-1))$. Set

$$\Omega_d'' := \bigcup_{d'\geq d} \mathcal{B}_{N(d'),d'}, \quad \Omega'' := \bigcap_d \Omega_d''$$

If $d' \geq d$ then $\alpha(d') \leq \alpha(d)$ and $r(N(d')) \leq r(N(d))$, due to $N(d) \leq N(d')$. Hence

$$\begin{aligned}\mathcal{H}^{\alpha(d)}_{r(N(d))}(\Omega_d'') &\leq \sum_{d'\geq d} \mathcal{H}^{\alpha(d)}_{r(d)}(\mathcal{B}_{N,d'}) \leq \sum_{d'\geq d} \mathcal{H}^{\alpha(d')}_{r(d')}(\mathcal{B}_{N,d'}) \\ &\leq \sum_{d'\geq d} \exp(-(\log\log N(d'))^B) \leq \sum_{d'\geq d} (d')^{-B} \lesssim d^{-B+1} \leq d^{-1}\end{aligned}$$

Therefore $\mathcal{H}^{\alpha(d)}_{r(N(d))}(\Omega'') \lesssim d^{-1}$ for any d. Since $\alpha(d), r(N(d)) \to 0$ with $d \to \infty$, the Hausdorff dimension of Ω'' is zero.

(d) Set $\Omega := \Omega' \cup \Omega''$. Then the Hausdorff dimension of Ω is zero. Assume that $\omega \in \mathbb{T}_{c,a} \setminus \Omega$. Then there exist N_1', d_1 such that $\omega \notin \hat{\Omega}_{N_1'}$, and $\omega \notin \mathcal{B}_{N(d),d}$ for any $d \leq d_1$. Pick arbitrary $d \geq d_1$ such that $N(d) \geq N_1'$. Set $N_1 := N(d)$. Then $\hat{\Omega}_{N_1} \subset \hat{\Omega}_{N_1'}$. Hence, $\omega \in \mathbb{T}_{c,a}\setminus(\hat{\Omega}_{N_1} \cup \mathcal{B}_{N_1,d})$. Due to (b) any given interval (E', E'') contains a subinterval (E_1', E_1'') such that $\Sigma_\omega \cap (E_1', E_1'') = \emptyset$ as claimed.

□

Appendix A. Polynomials, Resultants, and Algebraic Functions

We recall some basic facts on polynomials. They can be found in [Lan], chapter 5.

- A polynomial $f(x), x = (x_j, \dots, x_\nu)$, of ν variables $x_j \in \mathbb{R}, j - 1, 2 \dots n$ is called irreducible if there is no factorization
$$f(x) = g(x)h(x)$$
with $g(x), h(x)$ being non-constant polynomials.
- Each polynomials $f(x)$ can be factorized
$$f(x) = \prod_{j=1}^{m} f_j(x)$$
with $f_j(x)$ irreducible. This factorization is unique up to scalars.
- Let
$$f(x_1, \dots, x_\nu) = \sum_{j=0}^{k} a_j(x_2, \dots, x_\nu)x_1^j, \ \ \alpha_k \not\equiv 0$$
$$g(x_1, \dots, x_\nu) = \sum_{j=0}^{m} b_j(x_2, \dots, x_\nu)x_1^j, \ \ b_m \not\equiv 0$$
be two arbitrary polynomials of ν variables. The following determinant
$$\text{(A.1)} \qquad \operatorname{Res}(f, g) = \begin{vmatrix} \overbrace{\begin{matrix} a_k & 0 & \cdots \\ a_{k-1} & a_k & \cdots \\ a_{k-2} & a_{k-1} & \cdots \\ \cdots & \cdots & \cdots \\ a_0 & a_1 & \cdots \\ 0 & a_0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{matrix}}^{m} & \overbrace{\begin{matrix} b_m & 0 & \cdots & 0 \\ b_{m-1} & b_m & \cdots & \cdots \\ b_{m-2} & b_{m-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}}^{k} \end{vmatrix}$$
is called the resultant of f and g with respect to the first variable. For arbitrary variable x_j the resultant is defined similarly.
- Let $f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0$, $g(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_0$, $a_i, b_j \in \mathbb{C}$. Let ζ_i, $1 \le i \le k$ and η_j, $1 \le j \le m$ be the zeros of $f(z)$ and $g(z)$, respectively. The resultant of f and g satisfies
$$\operatorname{Res}(f, g) = \prod_{i,j} (\zeta_i - \eta_j)$$
The discriminant of the polynomial f is defined as
$$\text{(A.2)} \qquad \operatorname{disc} f = \prod_{i \ne j} \big(\zeta_i - \zeta_j\big) \ .$$
One has also
$$\operatorname{disc} f = (-1)^{n(n-1)/2} \operatorname{Res}(f, f') \ .$$
- Let $R(x_2, \dots, x_\nu)$ be as in (A.1). $R(x_2, \dots, x_\nu)$ is a polynomial $\deg(R) \le \deg(f) \deg(g)$. Recall that for
$$\text{(A.3)} \qquad f(x) = \sum_{\alpha} c_\alpha x^\alpha$$
The degree $\deg(f)$ is defined as $\max\{|\alpha| : c_\alpha \ne 0\}$ Here
$$x = (x_1, \dots, x_\nu), \ \alpha = (\alpha_1, \dots, \alpha_\nu), \ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_\nu^{\alpha_\nu}.$$

- The main property of the resultant is as follows

$$R(f,g) \equiv 0$$

If and only if

$$f = hf_1, \;\; g = hg_1$$

- In particular, if $f = \sum_{j=0}^{k} a_j(x_1,\dots,x_\nu)x_1^j,\; a_k(x_2,\dots,x_\nu) \not\equiv 0$ is irreducible, then

$$\mathrm{Res}(f,g) \equiv 0$$

for any g.

- Let $f(x,y), g(x,y)$ be polynomials of two variables, $x,y \in \mathbb{R}$. Bezout's Theorem says that if f or g is irreducible then the system

$$f(x,y) = 0, \quad g(x,y) = 0$$

has at most $m \cdot n$ solutions, $m = \deg f, n = \deg g$.

- Let $f(x,y)$ be a polynomial, and let $\varphi(x)$ be a function defined on some interval $[a,b]$ such that

$$f(x,\varphi(x)) = 0, \quad \text{for any} \quad x \in [a,b] \tag{A.4}$$

has $m(m-1)$ solution at most. If for some $x_0 \in [a,b]$

$$\partial_y f(x_0, y(x_0)) \neq 0$$

then $\varphi(x)$ is real analytic in some neighborhood of x_0,

$$\partial_x \varphi = -\partial_x f / \partial_y f$$

On the other hand if $\varphi(x)$ is real analytic on $[a,b]$ and obeys (A.3), then there exists an irreducible polynomial $f(x,y)$ such that

$$f_1(x,\varphi(x)) = 0 \qquad \text{for any} \qquad x \in [a,b]$$

for any $\mu \in \mathbb{R}$ the number of solutions of the equation

$$\partial_x \varphi = \mu$$

does not exceed $2m(m-1)$, where $m = \deg f$.

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